

Physics 139A Homework 5

Eric Reichwein
Department of Physics
University of California, Santa Cruz

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1 Problem 5.4

(a) If ψ_a and ψ_b are orthogonal, and both normalized, what is the constant A in Equation 5.10?

(b) If $\psi_a = \psi_b$, what is A ?

1.1 Solution to Problem 5.4 Part A

Equation 5.10 is

$$\psi_{\pm}(\vec{x}) = A[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)]$$

To determine A we normalize the wavefunction.

$$\begin{aligned} 1 &= \int |\psi_{\pm}(\vec{x})|^2 dx_1 dx_2 \\ &= \int A^2 [\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)]^2 dx_1 dx_2 \\ &= A^2 \int ([\psi_a(x_1)\psi_b(x_2)]^2 + [\psi_b(x_1)\psi_a(x_2)]^2 \pm 2\psi_b(x_1)\psi_a(x_2)\psi_a(x_1)\psi_b(x_2)) dx_1 dx_2 \\ &= A^2 \int \psi_a^2(x_1)\psi_b^2(x_2) dx_1 dx_2 + \int \psi_b^2(x_1)\psi_a^2(x_2) dx_1 dx_2 \pm \int 2\psi_b(x_1)\psi_a(x_2)\psi_a(x_1)\psi_b(x_2) dx_1 dx_2 \\ &= A^2 \int \psi_a^2(x_1) dx_1 \int \psi_b^2(x_2) dx_2 + \int \psi_b^2(x_1) dx_1 \int \psi_a^2(x_2) dx_2 \pm \int 2\psi_b(x_1)\psi_a(x_1) dx_1 \int \psi_a(x_2)\psi_b(x_2) dx_2 \\ &= A^2 [1 + 1 \pm 0] \\ &= 2A^2 \end{aligned} \tag{1}$$

Where we have used the orthonormality of the wavefunctions ($\int \psi_i \psi_j dx = \delta_{ij}$) to determine the values of the integrals. The constant is then just

$$A = \frac{1}{\sqrt{2}}$$

1.2 Solution to Problem 5.4 Part B

If $\psi_a = \psi_b = \psi$ then the only that changes from Part A is the last term is not zero.

$$\begin{aligned} 1 &= \int |\psi_{\pm}(\vec{x})|^2 dx_1 dx_2 \\ &= \int A^2 [\psi(x_1)\psi(x_2) \pm \psi(x_1)\psi(x_2)]^2 dx_1 dx_2 \\ &= A^2 \int ([\psi(x_1)\psi(x_2)]^2 + [\psi(x_1)\psi(x_2)]^2 \pm 2\psi(x_1)\psi(x_2)\psi(x_1)\psi(x_2)) dx_1 dx_2 \\ &= A^2 \int \psi^2(x_1)\psi^2(x_2) dx_1 dx_2 + \int \psi^2(x_1)\psi^2(x_2) dx_1 dx_2 \pm \int 2\psi(x_1)\psi(x_2)\psi(x_1)\psi(x_2) dx_1 dx_2 \\ &= A^2 \int \psi^2(x_1) dx_1 \int \psi^2(x_2) dx_2 + \int \psi^2(x_1) dx_1 \int \psi^2(x_2) dx_2 \pm \int 2\psi^2(x_1) dx_1 \int \psi^2(x_2) dx_2 \\ &= A^2 [1 + 1 \pm 2] \\ &= 4A^2 \end{aligned} \tag{2}$$

Where we have used the orthonormality of the wavefunctions ($\int \psi_i \psi_j dx = \delta_{ij}$) to determine the values of the integrals. We note that the bracketed term is either 4 or 0, but we re disregard the zero because that is the trivial case of no particles at all. This is due to the Pauli exclusion principle and the anti-symmetry of the spatial wavefunctions of two identical fermions. That being said the constant is

$$A = \frac{1}{2}$$

2 Problem 5.6

Imagine two noninteracting particles, each of mass m , in the infinite square well. If one is in the ψ_n state, and the other in state ψ_ℓ , evaluate $\langle (x_1 - x_2)^2 \rangle$, assuming

1. They are distinguishable particles
2. They are identical bosons
3. They are identical fermions

2.1 Solution to Problem 5.6 Part A

First we note that from Equation 5.18:

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

Now, using this definition and the properties of the infinite well wavefunctions (mainly orthogonality), we can readily calculate expectation values. If they are distinguishable then there wavefunction is

$$\psi(x_1, x_2) = \psi_n(x_1)\psi_\ell(x_2)$$

We will also need to know an integral which I have evaluated using Wolfram

Let us break this problem up into the two parts. First calculate $\langle x_n^2 \rangle$ (without loss of generality)

$$\begin{aligned}
\langle x_1^2 \rangle &= \int \psi_n(x_1)\psi_\ell(x_2)x_1^2\psi_n(x_1)\psi_\ell(x_2)dx_1dx_2 \\
\langle x_1^2 \rangle &= \int x_1^2\psi_n^2(x_1)dx_1 \int \psi_\ell^2(x_2)dx_2 \\
\langle x_1^2 \rangle &= \int x_1^2\psi_n^2(x_1)dx_1 \\
\langle x_1^2 \rangle &= \frac{2}{L} \int_0^L x_1^2 \sin^2(n\pi x_1/L)dx_1 \\
\langle x_1^2 \rangle &= \frac{2}{L} \left(\frac{L}{n\pi}\right)^3 \int_0^{n\pi} z^2 \sin^2(z)dz \\
\langle x_1^2 \rangle &= \frac{2}{L} \left(\frac{L}{n\pi}\right)^3 [4z^3 + (3 - 6z^2) \sin(2z) - 6z \cos(2z)]_0^{n\pi} \\
\langle x_1^2 \rangle &= \frac{2}{L} \left(\frac{L}{n\pi}\right)^3 \frac{1}{24} [(4(n\pi)^3 + (3 - 6(n\pi)^2) \sin(2n\pi) - 6n\pi \cos(2n\pi)) - (0 + (3 - 0)0 - 0 \cos(0))] \\
\langle x_1^2 \rangle &= \frac{1}{12L} \left(\frac{L}{n\pi}\right)^3 [4(n\pi)^3 - 6n\pi] \\
\boxed{\langle x_1^2 \rangle = L^2 \left[\frac{1}{3} - \frac{1}{2n^2\pi^2} \right]}
\end{aligned}$$

Now we can do the same thing to find the expectation value of x_2 except the answer is dependent on ℓ instead of n .

$$\boxed{\langle x_2^2 \rangle = L^2 \left[\frac{1}{3} - \frac{1}{2\ell^2\pi^2} \right]}$$

The last step is to determine the value of $\langle x_1x_2 \rangle$. We will do that explicitly

$$\begin{aligned}
2\langle x_1x_2 \rangle &= 2 \int \psi_n(x_1)\psi_\ell(x_2)x_1x_2\psi_n(x_1)\psi_\ell(x_2)dx_1dx_2 \\
&= 2 \int x_1\psi_n^2(x_1)dx_1 \int x_2\psi_\ell^2(x_2)dx_2 \\
&= \frac{4}{L} \int_0^L x_1 \sin^2(n\pi x_1/L)dx_1 \int_0^L x_2 \sin^2(\ell\pi x_2/L)dx_2 \\
&= 2 \frac{L}{2} \frac{L}{2} \\
&= \boxed{\frac{L^2}{2}}
\end{aligned} \tag{3}$$

Where every term in the integral (as seen in Figure 2) cancel except the first term. This is also what we expect to obtain because the particles average position should be in the center of the well.

Putting all of these together we get

$$\boxed{\langle (x_1 - x_2)^2 \rangle = L^2 \left[\frac{1}{6} - \frac{1}{2n^2\pi^2} - \frac{1}{2\ell^2\pi^2} \right]}$$

2.2 Solution to Problem 5.6 Part B

Now for identical bosons we will use Equation 5.21 with upper sign. Equation 5.21 is

$$\langle (x_1 - x_2)^2 \rangle_{\pm} = \langle x_1^2 \rangle_a + \langle x_2^2 \rangle_b - 2\langle x_1 \rangle_a \langle x_2 \rangle_b \pm |\langle x \rangle_{ab}|^2$$

However, we see the only difference between Part A and Part B (and Part C!) is the last term so all we have to do is calculate the last term and tack it onto the answer in Part A. We will need the following integral to help us calculate the last term of Equation 5.21. I evaluated it using Wolfram Alpha.


[Examples](#) [Random](#)

Indefinite integral:

[Step-by-step solution](#)

$$\int x \sin(ax) \sin(bx) dx = \frac{1}{2} \left(\frac{x \sin(x(a-b))}{a-b} - \frac{x \sin(x(a+b))}{a+b} + \frac{\cos(x(a-b))}{(a-b)^2} - \frac{\cos(x(a+b))}{(a+b)^2} \right) + \text{constant}$$

$$\begin{aligned} \langle x \rangle_{n\ell} &= \int \psi_n(x) \psi_\ell(x) x dx \\ &= \frac{2}{a} \int_0^L x \sin(n\pi x/L) \sin(\ell\pi x/L) dx \\ &= \frac{1}{L} \left(\left[\frac{1}{(n-\ell)\pi} \right]^2 (\cos((n-\ell)\pi) - 1) - \left[\frac{1}{(n+\ell)\pi} \right]^2 (\cos((n+\ell)\pi) - 1) \right) \\ &= \frac{L}{\pi^2} ((-1)^{n+\ell} - 1) \left[\frac{1}{(n-\ell)^2} - \frac{1}{(n+\ell)^2} \right] \end{aligned} \tag{4}$$

We can compactly write this as

$$\frac{L}{\pi^2} ((-1)^{n+\ell} - 1) \left[\frac{1}{(n-\ell)^2} - \frac{1}{(n+\ell)^2} \right] = \begin{cases} 0, & n + \ell = \text{even} \\ -\frac{8L n \ell}{\pi^2 (n^2 - \ell^2)^2}, & n + \ell = \text{odd} \end{cases}$$

Hence, the final answer is

$$\langle (x_1 - x_2)^2 \rangle = L^2 \left[\frac{1}{6} - \frac{1}{2n^2\pi^2} - \frac{1}{2\ell^2\pi^2} - \frac{128(n\ell)^2}{\pi^4(n^2 - \ell^2)^4} \right]$$

Where the last term only survives if the particles are in states with opposite parity, otherwise they have same expected separation as distinguishable particles.

2.3 Solution to Problem 5.6 Part C

From symmetry we can see immediately without going through any of the tedious calculations that for identical fermions we can just say that expected separation is

$$\langle (x_1 - x_2)^2 \rangle = L^2 \left[\frac{1}{6} - \frac{1}{2n^2\pi^2} - \frac{1}{2\ell^2\pi^2} + \frac{128(n\ell)^2}{\pi^4(n^2 - \ell^2)^4} \right]$$

Where the last term only survives if the particles are in states with opposite parity, otherwise they have same expected separation as distinguishable particles.

3 Problem 5.12

- Figure out the electron configurations for the first two rows of the Periodic Table, and check your results against Table 5.1.
- Figure out the corresponding total angular momenta, in the notation of Equation 5.34, for the first four elements. List all the possibilities for boron, carbon, and nitrogen.

3.1 Solution to Problem 5.12 Part A and B

The first two rows just fill the electron configuration up to the p orbital. Hence, we just fill the electrons in the orbitals sequentially. The superscript is $2S + 1$ where S is the total spin, the letter tells you the total angular momentum and the subscript tells you the total angular momentum and spin, $J = |L \pm S|$. For each value of $2S + 1$ we have to use each value of total angular momentum L to get all spectroscopic terms. Note that we can ignore the inner electrons because they are locked into singlet states of total angular momentum of zero and total spin of zero. The p -orbital electrons can be in either singlet or triplet configurations. The explanations for multiple spectroscopic states are laid out below along with a nice graphical representation of the orbitals.

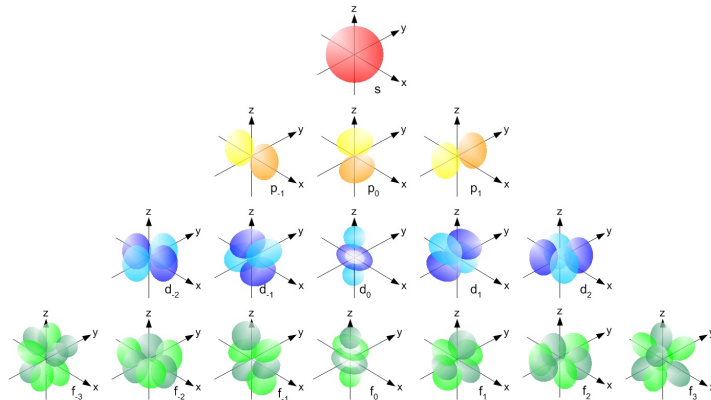


Figure 1: This is a graphical representation of the spdf single electron orbitals. Not of importance here, but very nice to visual what these orbitals correspond to.

Boron: Boron has one electron in its p -orbital. This means that each electron can have either $\ell = 0$ or $\ell = 1$ (corresponding to orbital angular or 0 or $\sqrt{2}\hbar$, respectively) and total spin of $S = 1/2 = \uparrow$ or \downarrow . For the spectroscopic term it can have two forms, one with total angular momentum of one or zero. This gives us $^2S_{1/2}$ or $^2P_{3/2}$.

Carbon: Carbon has two electrons in its p -orbital. This means that each electron can have either $\ell = 0$ or $\ell = 1$ (corresponding to orbital angular or 0 or $\sqrt{2}\hbar$, respectively). Since there is two electrons the total angular momentum L can take on the values $L = 0, 1, 2$ (because there is two electrons and each could have $\ell = 0, 1$). The total spin can take on values of $S = 0, 1$ corresponding to electrons that align like $1 = (\uparrow\uparrow, \downarrow\downarrow)$ or $0 = (\uparrow\downarrow)$. The total angular momentum J can be 0, 1, 2, or 3. This means there should be 10 total states that these atoms could have, and 10 different spectroscopic terms we need to determine. These states are: $^1S_0, ^3S_1, ^1P_1, ^3P_0, ^3P_1, ^3P_2, ^1D_2, ^3D_1, ^3D_2, ^3D_3$

Nitrogen: Nitrogen has three electrons in its p -orbital. This means that each electron can have either $\ell = 0$ or $\ell = 1$ (corresponding to orbital angular or 0 or $\sqrt{2}\hbar$, respectively). Since there is three electrons the total angular momentum L can take on the values $L = 0, 1, 2, 3$. The total spin can take on values of $S = 1/2, 3/2$ corresponding to electrons that align like $1/2 = (\uparrow\uparrow\downarrow, \downarrow\downarrow\uparrow)$ or $3/2 = (\uparrow\uparrow\uparrow, \downarrow\downarrow\downarrow)$. The total angular momentum will be all possible combinations of $J = |L \pm S|$. This would produce a combination of 14. These states are: $^2S_{1/2}, ^4S_{3/2}, ^2P_{1/2}, ^2P_{3/2}, ^4P_{5/2}, ^4P_{3/2}, ^4P_{1/2}, ^2D_{3/2}, ^2D_{5/2}, ^4D_{1/2}, ^4D_{3/2}, ^4D_{5/2}, ^4D_{7/2}, ^4D_{9/2}$.

I have summarized these results, along with the spectroscopic notations in the list seen below. Our results here are in accordance with the results in Table 5.1.

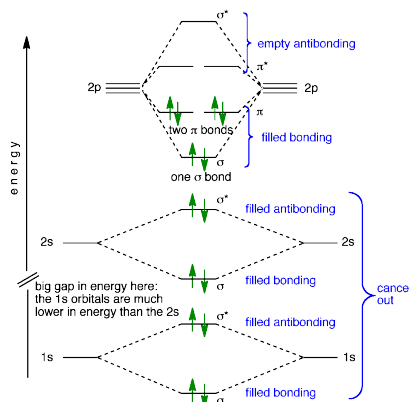


Figure 2: Here are the orbitals with electrons (including spin) for the nitrogen atom. Courtesy of chemtube3d.com.

¹The electrons could align in any permutation of these. As long the net spin is $\pm 1/2$ or $\pm 3/2$.

1. Hydrogen: $1s \implies^{2(1/2)+1} S_{1/2} =^2 S_{1/2}$
2. Helium: $1s^2 \implies^{2(0)+1} S_0 =^1 S_0$
3. Lithium: $1s^2 2s \implies^{2(1/2)+1} S_{1/2} =^2 S_{1/2}$
4. Beryllium: $1s^2 2s^2 \implies^{2(0)+1} S_{1/2} =^1 S_{1/2}$
5. Boron: $1s^2 2s^2 2p \implies^2 P_{1/2}, ^2 P_{3/2}$
6. Carbon: $1s^2 2s^2 2p^2 \implies^1 S_0, ^3 S_1, ^1 P_1, ^3 P_0, ^3 P_1, ^3 P_2, ^1 D_2, ^3 D_1, ^3 D_2, ^3 D_3$
7. Nitrogen: $1s^2 2s^2 2p^3 \implies^2 S_{1/2}, ^4 S_{3/2}, ^2 P_{1/2}, ^2 P_{3/2}, ^4 P_{5/2}, ^4 P_{3/2}, ^4 P_{1/2}, ^2 D_{3/2}, ^2 D_{5/2}, ^4 D_{1/2}, ^4 D_{3/2}, ^4 D_{5/2}, ^4 D_{7/2}, ^4 D_{9/2}, ^4 D_{5/2}, ^2 F_{5/2}, ^2 F_{7/2}, ^4 F_{3/2}, ^4 F_{5/2}, ^4 F_{7/2}, ^4 F_{9/2}$
8. Oxygen: $1s^2 2s^2 2p^4$
9. Fluorine: $1s^2 2s^2 2p^5$
10. Neon: $1s^2 2s^2 2p^6$

4 Problem 5.16

The density of copper is 8.96g/cm^3 , and its atomic weight is 63.5g/mole .

- (a) Calculate the Fermi energy for copper. Give answer in electron volts.
- (b) What is the corresponding electron velocity?
- (c) At what temperature would the characteristic thermal energy equal the Fermi, for copper?
- (d) Calculate the degeneracy pressure.

4.1 Solution to Problem 5.16 Part A

First we need to know how many free electrons copper has. For this we consult Table 5.1 in Griffiths where we see that copper is $(Ar)4s3d^{10}$ which means there is one extra electron in the $4s$ orbital and a completely filled $5d$ orbital². This means there is one extra electron able to conduct, or $q = 1$.

We also need the number of atoms per volume, N/V , to obtain the Fermi energy. To get this we just to divide the density of copper by the atomic weight, then multiply by Avogadro's number.

$$N/V = \frac{8.96\text{g/cm}^3}{63.5\text{g/mole}} \cdot 6.02 \times 10^{23}\text{molecules/mole} = 8.5 \times 10^{22}\text{e}^-/\text{cm}^3 = 8.5 \times 10^{28}\text{e}^-/\text{m}^3$$

Now we have everything needed to calculate the Fermi energy. We will skip the derivation as it is very straightforward and is easily found in any quantum mechanics or solid state textbook. We just make use of Equation 5.43

$$E_F = \frac{\hbar^2}{2m} \left(3 \frac{N}{V} q \pi^2 \right)^{2/3} = 1.128 \times 10^{-18}\text{J} = 7.04\text{eV}$$

where we have substituted the value of ρ of Equation 5.43 with its equivalent Nq/V .

²Remember that the screening due to inner electrons produces non-sequential filling of orbitals; since lower angular momentum states of higher energy orbitals can produce lower energy states!

4.2 Solution to Problem 5.16 Part B

To find the Fermi velocity we will just set the Fermi energy equal to the classical kinetic energy.

$$E_F = \frac{1}{2}mv^2 \longrightarrow v = \sqrt{\frac{2E_f}{m}} = \sqrt{\frac{2(7.04eV)}{0.511 \times 10^6 eV/c^2}} = 0.005c \approx 1.6 \times 10^6 m/s$$

We see that the velocity is much less than the speed of light, approximately half a percent of the speed of light, hence we can use the classical kinetic energy without having significant error in our results. We used the known value of the mass of an electron in terms of electron-volts (eV).

4.3 Solution to Problem 5.16 Part C

To determine the "temperature" of the copper material we will just set the Fermi energy equal to the thermal energy by making use of $E = k_b T$ where $k_b = 1.38 \times 10^{-23} J \cdot K = 8.62 \times 10^{-5} eV \cdot K$ is the Boltzmann constant.

$$E_F = k_b T \longrightarrow T_F = \frac{E_F}{k_b} = \frac{7.04eV}{8.62 \times 10^{-5} eV \cdot K} = 8.2 \times 10^5 K$$

4.4 Solution to Problem 5.16 Part D

The degeneracy pressure is easily calculated using Equation 5.46 in Griffiths

$$P = \frac{(3\pi^2)^{2/3} \hbar^2}{2m} (N/V)^{5/3} = 6.4 \times 10^{10} N/m^2$$

5 Problem 5.29

- Show that for bosons the chemical potential must always be less than the minimum allowed energy.
- In particular, for the ideal bose gas, $\mu(T) < 0$ for all T . Show that in this state $\mu(T)$ monotonically increases as T decreases, assuming N and V are held constant.
- Bose condensation crisis. Evaluate the integra, for $\mu = 0$, and obtain the formula for the critical temperature T_c at which this happens.
- Find the critical temperature for ${}^4\text{He}$. Its density at this temperature is $0.15g/cm^3$.

5.1 Solution to Problem 5.29 Part A

The chemical potential must have a minimum value if $n(\epsilon)$ is to remain positive (it has to remain positive because a negative value would be nonsense). This value is always positive as long as $e^{(\epsilon-\mu)/k_b T} > 1$, hence

$$\begin{aligned} e^{(\epsilon-\mu)/k_b T} &> 1 \\ \ln(e^{(\epsilon-\mu)/k_b T}) &> \ln(1) \\ (\epsilon - \mu)/k_b T &> 0 \\ \boxed{\epsilon > \mu} \end{aligned}$$

This means that the chemical potential must always be less than the energy of the state in question.

5.2 Solution to Problem 5.29 Part B

Looking at equation 5.108

$$N = \frac{V}{2\pi^2} \int_0^\infty \frac{k^2}{e^{[(\hbar^2 k^2/2m) - \mu]/k_b T} - 1} dk$$

We see that the exponential should stay constant for N and V to stay constant. If T starts decreasing the denominator of the exponential will be getting smaller. This means the numerator should get smaller too, but the first term (energy term is relatively constant) hence μ should increase monotonically to decrease the entire numerator just as much as the denominator decreases. But not that since μ is always less than the energy $\epsilon = \frac{\hbar^2 k^2}{2m}$ and μ is less than zero it will be increasing but always negative. QED.

5.3 Solution to Problem 5.29 Part C

If we let $\mu = 0$, then Equation 5.108 becomes

$$N/V = \frac{1}{2\pi^2} \int_0^\infty \frac{k^2}{e^{\hbar^2/(2mk_bT)k^2} - 1} dk$$

Now if we use the substitution $x = \frac{\hbar^2}{(2mk_bT)}k^2$ with $dx = \frac{\hbar^2}{(2mk_bT)}2kdk$ and $k = \sqrt{\frac{2mk_bT}{\hbar^2}}\sqrt{x}$ then our integral becomes

$$N/V = \frac{1}{2\pi^2} \int_0^\infty \frac{\sqrt{\frac{2mk_bT}{\hbar^2}}\sqrt{x}}{e^x - 1} \frac{dx}{\frac{\hbar^2}{(4mk_bT)}} = \frac{1}{4\pi^2} \left(\frac{2mk_bT}{\hbar^2} \right)^{3/2} \int \frac{\sqrt{x}}{e^x - 1} dx$$

Where we know this integral as

$$\int \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s)\zeta(s)$$

For our integral $s = 3/2$ hence our integral becomes

$$N/V = \frac{1}{4\pi^2} \left(\frac{2mk_bT}{\hbar^2} \right)^{3/2} \Gamma(3/2)\zeta(3/2) = \frac{1}{4\pi^2} \left(\frac{2mk_bT}{\hbar^2} \right)^{3/2} \frac{\sqrt{\pi}}{2} 2.61$$

Now we know that N and V are constant and that this is a critical value when Bose condensation occurs. Hence we can find the critical temperature

$$N/V = \frac{1}{4\pi^2} \left(\frac{2mk_bT}{\hbar^2} \right)^{3/2} \frac{\sqrt{\pi}}{2} 2.61 \rightarrow \boxed{T_c = \frac{2\pi\hbar^2}{mk_b} \left(\frac{N}{2.61V} \right)^{2/3}}$$

5.4 Solution to Problem 5.29 Part D

For ${}^4\text{He}$ we use the given density and the same technique in problem 5.16 Part A to find the number electrons per unit volume. We get

$$N/V = \frac{0.15g/cm^3}{4g/mole} 6.02 \times 10^{23} \text{ molecules/mole} = 2.26 \times 10^{28} e^-/m^3$$

Now plugging this value into the result of Part C we get

$$\boxed{T_c = \frac{2\pi\hbar^2}{mk_b} \left(\frac{N}{2.61V} \right)^{2/3} = \frac{2\pi\hbar^2}{mk_b} \left(\frac{1}{2.61} (2.26 \times 10^{28} e^-/m^3) \right)^{2/3} \approx 3.1K}$$

6 Problem 5.35

Certain cold stars (called white dwarfs) are stabilized against gravitational collapse by the degeneracy pressure of their electrons (Equation 5.46). Assuming constant density, the radius R of such an object can be calculated as follows:

- Write the total electron energy (Equation 5.45) in terms of the radius, the number of nucleons N , the number of electrons per nucleon q , and the mass m .
- Look up, or calculate, the gravitational energy of a uniformly dense sphere.
- Find the radius which total energy is minimum.
- Determine the radius of a white dwarf with mass of the sun.
- Determine the Fermi energy for the white dwarf and compare it with rest energy of an electron.

6.1 Solution to Problem 5.35 Part A

Starting with the degeneracy pressure, Equation 5.46, we can solve for E_{tot} easily.

$$P = \frac{2E_{tot}}{3V} = \frac{(3\pi^2)^{2/3}\hbar^2}{2m}(Nq/V)^{5/3}$$

Rearranging this equation we get

$$E_{tot} = \frac{3V}{2} \frac{(3\pi^2)^{2/3}\hbar^2}{2m}(Nq/V)^{5/3} = \frac{3}{2V^{2/3}} \frac{(3\pi^2)^{2/3}\hbar^2}{2m}(Nq)^{5/3}$$

We know everything except for V , the volume, but we know that the white dwarf is a sphere hence the volume is $V = (4/3)\pi R^3$. Our expression for E_{tot} becomes

$$E_{tot} = \frac{3}{2V^{2/3}} \frac{(3\pi^2)^{2/3}\hbar^2}{2m}(Nq)^{5/3} = \frac{3}{2(4\pi/3)^{2/3}R^2} \frac{(3\pi^2)^{2/3}\hbar^2}{2m}(Nq)^{5/3} = \frac{3(9\pi)^{2/3}\hbar^2}{mR^2} \left(\frac{Nq}{4}\right)^{5/3}$$

6.2 Solution to Problem 5.35 Part B

As the problem asks we shall look up or calculate the energy of a uniform dense sphere. I chose the lazy route and looked it up on Wikipedia. The derivation is shown below

Derivation for a uniform sphere [\[edit\]](#)

The gravitational binding energy of a sphere is found by imagining that it is pulled apart by successively moving spherical shells to infinity, the outermost first, and finding the energy needed for that.

If we assume a constant density ρ then the masses of a shell and the sphere inside it are:

$$m_{\text{shell}} = 4\pi r^2 \rho dr \quad \text{and} \quad m_{\text{interior}} = \frac{4}{3}\pi r^3 \rho$$

The required energy for a shell is the negative of the gravitational potential energy:

$$dU = -G \frac{m_{\text{shell}} m_{\text{interior}}}{r}$$

Integrating over all shells we get:

$$U = -G \int_0^R \frac{(4\pi r^2 \rho) (\frac{4}{3}\pi r^3 \rho)}{r} dr = -G \frac{16}{15} \pi^2 \rho^2 R^5$$

Remembering that ρ is simply equal to the mass of the whole divided by its volume for objects with uniform density we get:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

And finally, plugging this in to our result we get:

$$U = -G \frac{16}{15} \pi^2 R^5 \left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 = \frac{-3GM^2}{5R}$$

Figure 3: This is the derivation given on the Wikipedia page for Gravitational Binding Energy. The link is <http://en.wikipedia.org/wiki/Gravitational-binding-energy>.

Hence, after one Google search we have come to the conclusion that the gravitational binding energy is just

$$E_{\text{bind}} = -\frac{3GM_{\text{total}}^2}{5R} = -\frac{3GN^2M^2}{5R}$$

Where the total mass M_{total} is just the mass of a nucleon times the number of nucleons, $M_{\text{total}} = NM$.

6.3 Solution to Problem 5.35 Part C

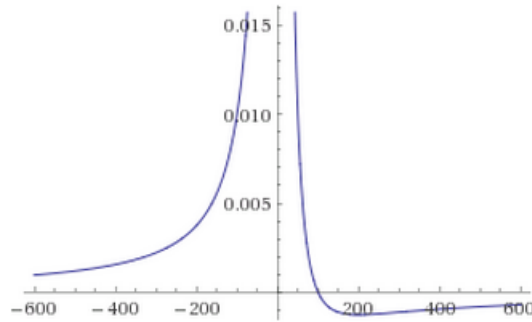
The total energy E_T of the white dwarf will be the total electron energy plus the total gravitational binding energy

$$E_T = E_{\text{tot}} + E_{\text{bind}} = \frac{\alpha}{R^2} - \frac{\beta}{R}$$

Where

$$\alpha = \frac{3(9\pi)^{2/3}\hbar^2}{m} \left(\frac{Nq}{4}\right)^{5/3} \quad \text{and} \quad \beta = \frac{3GN^2M^2}{5}$$

To find the radius that minimizes the total energy we must find the radius that makes the derivative of the total energy function equal to zero. I have graphed the total energy function below.



Taking the derivative of the total energy and setting it equal to zero we find that

$$\frac{dE_T}{dR} = -2\frac{\alpha}{R^3} - \left(-\frac{\beta}{R^2}\right) = 0 \rightarrow \beta R^3 = 2\alpha R^2 \rightarrow R = \frac{2\alpha}{\beta}$$

Plugging in the values for the constants α and β we get

$$R = \frac{2\alpha}{\beta} = \frac{2 \frac{3(9\pi)^{2/3}\hbar^2}{m} \left(\frac{Nq}{4}\right)^{5/3}}{\frac{3GN^2M^2}{5}} = \frac{7.6 \times 10^{25}m}{N^{1/3}}$$

6.4 Solution to Problem 5.35 Part D

To find the radius of a white dwarf with the mass of the sun all we need to do is calculate the total number of nucleons in the sun. We know the total mass of the sun and the mass of one nucleon hence the total number of nucleons is just

$$N = \frac{M_{sun}}{M_{nucleon}} = \frac{1.99 \times 10^{30}kg}{1.67 \times 10^{-27}kg/nucleon} = 1.2 \times 10^{57}nucleon \rightarrow \frac{1}{N^{1/3}} = 9.45 \times 10^{-20}$$

Plugging this value into our result of Part C we get

$$R = (7.6 \times 10^{25}m)(9.45 \times 10^{-20}) = 7.2 \times 10^5m$$

6.5 Solution to Problem 5.35 Part E

To find the Fermi energy we just use Equation 5.43 with our known values.

$$E_F = \frac{\hbar^2}{2m} \left(3\frac{N}{V}\pi^2q\right)^{2/3} = \frac{\hbar^2}{2mR^2} \left(\frac{9N\pi}{4}q\right)^{2/3} = 3 \times 10^{-14}J = 1.9MeV$$

Comparing this value to the rest mass energy of the electron ($0.511MeV/c^2$) we see that the electrons have nearly four times the rest mass energy hence would have a gamma factor of four. A gamma factor of four corresponds to a velocity of $v_f \approx \frac{\sqrt{15}}{4}c$ which is definitely relativistic.