Physics 139A Homework 4

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1 Problem 4.9

A particle of mass m is placed in a finite spherical well:

$$V(x) = \begin{cases} 0, & r > a \\ -V_0, & r \leq a \end{cases}$$

Solve the radial equation for the for the ground state case.

1.1 Solution to Problem 4.9

There are two regions for the finite spherical well. The region outside the well, $r > a$, and the region that is inside the well, $r \leq a$. Let us write out Schrodinger's equation for both inside and outside the well. Note that since the particle is bound to within the well its overall energy is $E < 0$. I will right E as the magnitude and carry around the negative value.

First note that $\ell = 0$ hence we don't have an "effective" potential to account for.

Inside the well:

$$\begin{aligned} \frac{d^2u}{dr^2} &= \left[\frac{\ell(\ell+1)}{r^2} - k^2 - V_0\right] u \\ \frac{d^2u}{dr^2} &= \left[-k^2 - V_0\right] u \\ \frac{d^2u}{dr^2} &= -\left[k^2 + V_0\right] u \end{aligned}$$

This has solution

$$u(r) = Asin(\kappa r) + Bcos(\kappa r)$$

Where $\kappa^2 = k^2 + V_0 = \frac{2m(E+V_0)}{\hbar^2}$

Outside the well:

$$\begin{aligned} \frac{d^2u}{dr^2} &= \left[\frac{\ell(\ell+1)}{r^2} - k^2\right] u \\ \frac{d^2u}{dr^2} &= \left[-k^2\right] u \\ \frac{d^2u}{dr^2} &= -k^2 u \end{aligned}$$

This has solution

$$u(r) = Ce^{-kr} + De^{kr}$$

Where $k^2 = \frac{-2mE}{\hbar^2}$

Now physically the wavefunction must go to zero when outside the well, because it is a bound state. The general solution for this is This must mean that for $r < -a$ then $A = 0$, and for $x > a$ then $B = 0$. Since $\ell = 0$ we are do not have to deal with with general spherical bessel and neumann functions. We know that $u(r) = R(r)/r$ so let us examine the boundary conditions closely. At $r \longrightarrow 0$ we have $u(r) = A\sin(\kappa r)/r + B\cos(\kappa r)/r$. We know from Taylor expansion that the cosine term is singular at $r = 0$, but the sine over r term goes to 1, therefore $B = 0$. Outside the well we have just a decaying exponential (the exponential term dominates the $1/r$) hence $D = 0$. The final solution is

$$V(x) = \begin{cases} A\sin(\kappa r)/r, & r < a \\ Ce^{-kr}, & r > a \end{cases}$$

Now that we have the general wavefunction we must now impose boundary conditions. At the walls of the well the wavefunction should be smooth and continuous.

Continuous Condition:

$$\begin{aligned} R_{in}(a) =& R_{out}(a) \\ A\sin(\kappa a)/a =& De^{-ka} \end{aligned}$$

Smoothness Condition:

$$\begin{aligned} \left.\frac{\partial R_{in}(r)}{\partial r}\right|_{r=a} =& \left.\frac{\partial R_{out}(r)}{\partial r}\right|_{x=r} \\ \kappa Acos(\kappa a)/a =& -kDe^{-ka} \end{aligned}$$

Now to get rid of the constants and the exponential we will divide both equations by eachother. This will tell us when for what values of κ and k there will be stationary states. The condition that tells us the quantization is

$$\kappa cot(\kappa a) = -k \longrightarrow -\frac{k}{k} = cot(ka)$$

By squaring the separation constants (k and κ) and adding them we get

$$k^2 + \kappa^2 = \frac{2m(V_0 - E)}{\hbar^2} + \frac{2mE}{\hbar^2} = \frac{2mV_0}{\hbar^2} \longrightarrow \frac{\kappa}{k} = \sqrt{\frac{2mV_0}{k^2\hbar^2} - 1}$$

Now defining $z = ka$ and $z_0 = \frac{a\sqrt{2mV_0}}{\hbar}$ we can rewrite the last expressions as

$$-\frac{\kappa}{k} = -\sqrt{\frac{2mV_0}{k^2\hbar^2} - 1} = cot(ka) \longrightarrow \sqrt{\frac{z_0^2}{z^2} - 1} = -cot(z)$$

The condition for a bound state is when $z_0 < z$ and when $z < \pi/2$. Putting these together we get

$$\frac{a\sqrt{2mV_0}}{\hbar} < \frac{\pi}{2} \longrightarrow \boxed{V_0 a^2 < \frac{\pi^2\hbar^2}{8m}}$$

2 Problem 4.13

(a) Find $\langle r \rangle$ and $\langle r^2 \rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr Radius.

(b) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.

(c) Find $\langle x^2 \rangle$ in the state $n = 2$, $\ell = 1$, $m = 1$.

2.1 Solution to Problem 4.13 Part A

We start with the definition of the expectation value of r

$$\begin{aligned}
\langle r \rangle &= \int \Psi_{100}^* r \Psi_{100} r^2 \sin\theta dr d\theta d\phi \\
&= \frac{1}{\pi a^3} \int 4\pi r^3 e^{-2r/a} dr \\
&= \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} dr \\
&= \frac{4}{a^3} \int_0^\infty \frac{a^3}{8} u^3 e^{-u} \frac{a}{2} du \\
&= \frac{4}{a^3} \frac{a^4}{16} \left[e^{-u}(-r^3 - 3r^2 - 6r - 6) \right]_0^\infty \\
&= \frac{a}{4} [6] \\
&= \boxed{\frac{2a}{3}}
\end{aligned} \quad (1)$$

Where I have evaluated the integral using Wolfram after simplifying it by u-substitution. Now we will evaluate the expectation value of the radial distance squared

$$
\begin{aligned}
\langle r^2 \rangle &= \int \Psi_{100}^* r^2 \Psi_{100} r^2 \sin\theta dr d\theta d\phi \\
&= \frac{1}{\pi a^3} \int 4\pi r^4 e^{-2r/a} dr \\
&= \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} dr \\
&= \frac{4}{a^3} \int_0^\infty \frac{a^4}{16} u^3 e^{-u} \frac{a}{2} du \\
&= \frac{4}{a^3} \frac{a^5}{32} \left[e^{-u}(-r^4 - 4r^3 - 12r^2 - 24r - 24) \right]_0^\infty \\
&= \frac{a^2}{8} [24] \\
&= \boxed{3a}
\end{aligned}
\tag{2}
$$

2.2 Solution to Problem 4.13 Part B

The expectation value of x of the hydrogen wavefunction will be found the same way as r in part A, except that $r^2 = x^2 + y^2 + z^2$.

$$
\begin{aligned}
\langle r \rangle &= \int \Psi_{100}^* x \Psi_{100} dx dy dz \langle r \rangle \\
&= \frac{1}{\pi a^3} \left[\int x e^{-2\sqrt{x^2+y^2+z^2}/a} dx \right] dy dz \boxed{\langle r \rangle = 0}
\end{aligned}
\tag{3}
$$

This integral is zero because we are integrating an odd function over a symmetric region. For x^2 we first notice that x is an arbitrary direction. This means x, y and z have equal weights. Therefore, we can say that the expectation value of x^2 is just one third of the expectation value of r^2. We also see that the expectation value of x^2 does not produce an odd function hence it will not be zero.

$$
\boxed{\langle x^2 \rangle = \frac{1}{3} \langle r^2 \rangle = \frac{1}{3} 3a^2 = a^2}
$$

2.3 Solution to Problem 4.13 Part C

I looked up the wavefunction of an electron in this state on http://panda.unm.edu/Courses/Finley/P262/Hydrogen/WaveFcns.html. It is given as

$$
\frac{1}{8\sqrt{\pi}} \frac{r}{a^{3/2}} \sin\theta e^{-r/2a} e^{i\phi}
$$

For the hydrogen in the $n = 2, \ell = 1, m = 1$ state we have to use $x = r \sin theta \cos\phi$. Now we just integrate with respect to x over all space with our definition of x in spherical polar coordinates.

$$
\begin{aligned}
\langle x^2 \rangle &= \int \Psi_{211}^* x^2 \Psi_{211} r^2 \sin\theta dr d\theta d\phi \\
&= \int |\Psi_{211}^*|^2 (r \sin\theta \cos\phi)^2 r^2 \sin\theta dr d\theta d\phi \\
&= \frac{1}{64\pi a^5} \int r^2 e^{-r/a} \sin^2\theta)(r^2 \sin^2\theta \cos^2\phi r^2 \sin\theta dr d\theta d\phi \\
&= \frac{1}{64\pi a^5} \int_0^\infty r^6 e^{-r/a} dr \int \sin^5\theta d\theta \int \cos^2\phi d\phi \\
&= \frac{1}{64\pi a^5} \left[a^7 6! \right] \left[\frac{16}{15} \right] [\pi] \\
&= \frac{1}{64\pi a^5} \left[720 a^7 \right] \frac{16\pi}{15} \\
&= \boxed{12a^2}
\end{aligned}
\tag{4}
$$

3 Problem 4.16

Determine the Bohr energies $E_n(Z)$, the binding energy $E_1(Z)$, the Bohr radius $a(Z)$, and the Rydberg constant $R(Z)$ for a hydrogenic atom. Where in the electromagnetic spectrum would the Lyman series fall, for $Z = 2$ and $Z = 3$?

3.1 Solution to Problem 4.16

For the hydrogen atom the Bohr energies are

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$

We see that this has an e^4 dependence. However, we need to see where the inner e^2 came from. It comes from Coulomb potential between a proton and the electron. For Hydrogenic atoms this Coulomb potential is $(Ze)(e) = Ze^2$. Hence the Bohr energies are

$$\boxed{E_n(Z) = -\left[\frac{m}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = -\frac{E_n}{Z^2}}$$

For the binding energy $E_1(Z)$ we just evaluate all the terms. However, we can factor out our Z^2 and we are just left with the binding energy of hydrogen, $-13.6eV$, times the Z^2.

$$\boxed{E_1(Z) = -Z^2(13.6eV) = -Z^2E_1}$$

We use the same analysis as finding the Bohr energy of hydrogenic atoms as to find the Bohr radius. Hence, we just change $e^2 \longrightarrow Ze^2$ of equation 4.72.

$$\boxed{a(Z) = \frac{4\pi\epsilon_0\hbar^2}{me^2}\frac{1}{Z} = \frac{a}{Z}}$$

The Rydberg constant is

$$\boxed{R(Z) = \frac{m}{4\pi c\hbar^3}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2 = Z^2R}$$

Where R is the Rydberg constant of the Hydrogen atom. For Lyman series the resulting electromagnetic spectrum will be characterized by the energies

$$\Delta E_n(Z) = -Z^2(13.6eV)\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$$

For $Z = 2$ the resulting electromagnetic radiation emitted for the Lyman series is

$$\Delta E_{2\to1}(2) = -4(13.6eV)\left(\frac{1}{2^2} - 1\right) = -4(13.6eV)\frac{-3}{4} = 40.8eV$$

This is the lowest energy photon emitted. The highest energy photon emitted would be when $n_i = \infty$ which would just give $E_{max} = 4(13.6eV) = 54.4eV$

This range of energies can be converted to wavelengths using $\lambda = \frac{1240eV\cdot nm}{E}$. The corresponding range of wavelengths are $30.4nm$ to $22.8nm$ which would put the radiation in the extreme ultraviolet range.

For the $Z = 3$ case we have

$$E_{2\to1}(3) = -9(13.6eV)\left(\frac{1}{2} - 1\right) = 9(13.6eV)\frac{3}{4} = 91.8eV$$

Once again this is the lowest energy photon emitted. All other transitions would produce light of higher frequency. The highest frequency of light would be when $n_i \longrightarrow \infty$.

$$E_{\infty\to1}(3) = 9(13.6eV) = 122.4eV$$

This corresponds to a range of wavelengths of $13.5nm$ to $10.1nm$ where both of these lay in the extreme ultraviolet range, almost x-ray range.

4 Problem 4.17

Consider the earth-sun system as a gravitational analog to the hydrogen atom.

(a) What is the potential energy function?

(b) What is the "Bohr radius." a_g, for this system?

(c) Write down the gravitational "Bohr formula", and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_0, show that $n = \sqrt{r_0/a_g}$. From thus, estimate the quantum number n of the earth.

4.1 Solution to Problem 4.17 Part A

The potential energy of the system is purely gravitational, unlike the hydrogen atom which is purely electrostatic. Hence we have to replace the Coulomb potential with a gravitational potential.

$$\boxed{V_{hydrogen}(r) = -k\frac{e^2}{r} \longrightarrow V_{solar}(r) = -G\frac{Mm}{r}}$$

Where G is a constant, m is the mass of the earth and M is the mass of the sun.

4.2 Solution to Problem 4.17 Part B

All we need to do here is replace e^2 with Mm and $k = 1/(4\pi\epsilon_0)$ with G . Using this substitution with equation 4.72 we get

$$\boxed{a_g = \frac{\hbar^2}{Gm^2M}} \approx 2.3410^{-138}m$$

4.3 Solution to Problem 4.17 Part C

The Bohr formula is given as equation 4.70,

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} \longrightarrow E_n = -\left[\frac{m}{2\hbar^2}\left(GMm\right)^2\right]\frac{1}{n^2}$$

Now the energy of a circular orbit planet is just

$$E_p = -\frac{GMm}{2r_0}$$

which was shown in numerous times (including deriving the Bohr Formula) by relating the force to the centripetal acceleration and re-writing it as the kinetic energy. Now setting both of these equal we get

$$\begin{aligned}
E_p =& E_n \\
-\frac{GMm}{2r_0} =& -\left[\frac{m}{2\hbar^2}\left(GMm\right)^2\right]\frac{1}{n^2} \\
\frac{1}{2r_0} =& GMm\left[\frac{m}{2\hbar^2}\right]\frac{1}{n^2} \\
n^2 =& r_0\frac{GMm^2}{\hbar^2} \\
&\boxed{n = \sqrt{r_0\frac{GMm^2}{\hbar^2}} = \sqrt{r_0/a_g}}
\end{aligned}$$

Plugging in the numbers we get an approximation

$$n = \sqrt{r_0/a_g} = \sqrt{\frac{1.514 \times 10^{11}m}{2.34 \times 10^{-138}m}} \approx \boxed{6 \times 10^{74}}$$

4.4 Solution to Problem 4.17 Part D

The energy of a transition from $n+1$ to n state is given by

$$\Delta E_g = -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right)$$

Here we have n extremely large so that in this form we could not calculate the change in energy due to a change in state. We must use binomial theorem to simplify this result to something calculable. We must get the expression of n in terms of $(1 \pm x)^m$ where $x << 1$ for us to be able to binomially expand it out [1]. Starting with the $1/(n-1)^2$ term we factor out a n resulting in

$$(n+1)^{-2} = n^{-2}(1+1/n)^{-2} \approx n^{-2}(1-2/n)$$

Plugging this back in we get

$$\Delta E_g \approx -\left[\frac{m}{2\hbar^2}(GMm)^2\right]\left(\frac{1}{n^2}(-2/n+1) - \frac{1}{n^2}\right) = \left[\frac{m}{2\hbar^2}(GMm)^2\right]\frac{1}{n^2}((-2/n+1)-1) = -\left[\frac{m}{\hbar^2}(GMm)^2\right]\frac{1}{n^3}$$

Now by plugging in numbers for the variables we get

$$\boxed{\Delta g \approx 2.1 \times 10^{-41} J = 1.3 \times 10^{-22} eV}$$

And the corresponding wavelength of a photon emitted is given by

$$\boxed{\lambda = \frac{hc}{E} = \frac{1240 ev \cdot nm}{E} = \frac{1240 ev \cdot nm}{2.1 \times 10^{-22}} = 9.55 \times 10^{24} nm \approx 1 Lightyear}$$

This is a remarkable result because the wavelength of the photon emitted due to a transition on earth between two preceding states is equivalent to the length light travels in one revolution of earths orbit.

5 Problem 4.24

Two particles of mass m are attached to the ends of a massless rigid rod of length a. The system is free rotate in three dimensions about the center (but the center point itself is fixed).

(a) Show that the allowed energies of this rigid rotor are

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2} \qquad for\ n = 0, 1, 2...$$

(b) What are the normalized eigenfunctions for this system? What is the degeneracy of the nth energy level?

5.1 Solution to Problem 4.24 Part A

The classical energy is just the sum of the kinetic and potential energies. However, there is no potential energy in this system. Since the rod can rotate in three directions there will be three contributions to the kinetic energy for each dimension. The moment of inertia of one ball is mr^2 (the 2 is for the two masses). Also note that $L = I\omega$.

$$\begin{aligned}
E =& KE_x + KE_y + KE_z \\
E =& \frac{1}{2}I\omega_x^2 + \frac{1}{2}I\omega_y^2 + \frac{1}{2}I\omega_z^2 \\
E =& \frac{1}{2I}I^2\omega_x^2 + \frac{1}{2I}I^2\omega_y^2 + \frac{1}{2I}I^2\omega_z^2 \\
E =& \frac{1}{2I}L_x^2 + \frac{1}{2I}L_y^2 + \frac{1}{2I}L_z^2 \\
E =& \frac{1}{2I}\left(L_x^2 + L_y^2 + L_z^2\right) \\
E =& \frac{1}{2ma^2}\left(L_x^2 + L_y^2 + L_z^2\right) \\
E =& \frac{1}{2ma^2}L^2
\end{aligned}$$

[1] The first few terms of the binomial expansion is $(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2 + ...$

Here is where we make the transition from classical to quantum mechanical. We replace the classical angular momentum squared by the quantized angular momentum squared ($L^2 \rightarrow \hat{L}^2 = \hbar^2\ell(\ell+1)$).

$$E = \frac{1}{2ma^2}L^2 \longrightarrow E_n = \frac{1}{2ma^2}\hbar^2\ell(\ell+1)$$

But now since there is two balls we multiply this answer by two to get the total quantized energy.

$$\boxed{E_n = 2\frac{\hbar^2\ell(\ell+1)}{2ma^2} = \frac{\hbar^2\ell(\ell+1)}{ma^2}}$$

5.2 Solution to Problem 4.24 Part B

The only possible normalizable eigenfunctions for a rigid rotor would have to be purely angular dependence and give us energy eigenvalues proportional to $\ell(\ell+1)$. The only functions these could be are the spherical harmonics.

$$\boxed{\Psi(\theta, \phi) = Y_{\ell'}^m(\theta, \phi)}$$

Where ℓ' is the angular momentum quantum number and ℓ is the energy. This was sloppy notation of me due to the fact I used ℓ as the principle quantum number instead n in Part A. Eigenfunctions of spherical harmonics have a degeneracy of $\boxed{2\ell+1}$.

6 Problem 4.33

An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B_0\cos(\omega t)\hat{k}$$

(a) Construct the Hamiltonian matrix for this system.

(b) The electron starts out (at $t = 0$) in the spin-up state with respect to the x-axis (that is: $\chi(0) = \chi_+^{(x)}$). Determine the $\chi(t)$ at any subsequent time.

(c) Find the probability of getting $-\hbar/2$, if you measure S_x.

(d) What is the minimum field (B_0) required to force a complete flip in S_x.

6.1 Solution to Problem 4.33 Part A

Since the electron is at rest there is no kinetic energy term. Hence the only contribution to the Hamiltonian is the interaction between the magnetic field and the spin of the electron. According to equation 4.158 the Hamiltonian is

$$\boxed{H = -\gamma\mathbf{B}\cdot\mathbf{S} = \gamma\frac{\hbar B_0\cos(\omega t)}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

Where the I only used $\mathbf{S}_z$ because the magnetic field is entirely in the $\hat{z}$ direction. And therefore the dot product only gives a the z component of the Pauli spin vector.

6.2 Solution to Problem 4.33 Part B

We start with the time-dependent Schrodinger equation in the form of equation 4.162

$$i\hbar\frac{\partial\chi}{\partial t} = \mathbf{H}\chi \longrightarrow i\hbar\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \gamma\frac{\hbar B_0\cos(\omega t)}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

To solve this system of equations we first just solve alpha dependent equation, then the beta dependent equation. We can do this because the equations are completely independent (no cross terms).

$$\begin{aligned}
i\hbar\frac{d\alpha}{dt} =& \gamma\frac{\hbar B_0\cos(\omega t)}{2}\alpha \\
i\frac{d\alpha}{\alpha} =& \gamma\frac{B_0\cos(\omega t)}{2}dt \\
i\int^{\alpha}\frac{d\alpha}{\alpha} =& \gamma\frac{B_0}{2}\int^{t}\cos(\omega t)dt \\
i\ln(\alpha) =& \gamma\frac{B_0}{2\omega}\sin(\omega t)+C \\
\ln(\alpha) =& -i\gamma\frac{B_0}{2\omega}\sin(\omega t)-iC \\
\alpha =& Ae^{-i\gamma\frac{B_0}{2\omega}\sin(\omega t)}
\end{aligned}$$

Where $A = e^{-iC}$. We use our initial conditions to solve for A. At $t = 0$ then $\alpha = 1/\sqrt{2}$ hence $A = 1/\sqrt{2}$.

$$\boxed{\alpha = \frac{1}{\sqrt{2}}e^{-i\gamma\frac{B_0}{2\omega}\sin(\omega t)}}$$

Then to solve for beta we do the same thing

$$\begin{aligned}
i\hbar\frac{d\beta}{dt} =& -\gamma\frac{\hbar B_0\cos(\omega t)}{2}\beta \\
i\frac{d\beta}{\beta} =& -\gamma\frac{B_0\cos(\omega t)}{2}dt \\
i\int^{\beta}\frac{d\beta}{\beta} =& -\gamma\frac{B_0}{2}\int^{t}\cos(\omega t)dt \\
i\ln(\beta) =& \gamma\frac{B_0}{2\omega}\sin(\omega t)+C \\
\ln(\beta) =& -i\gamma\frac{B_0}{2\omega}\sin(\omega t)-iC \\
\beta =& Be^{-i\gamma\frac{B_0}{2\omega}\sin(\omega t)}
\end{aligned}$$

Where $B = e^{-iC}$. We use our initial conditions to solve for B. At $t = 0$ then $\beta = 1/\sqrt{2}$ hence $B = 1/\sqrt{2}$.

$$\boxed{\beta = \frac{1}{\sqrt{2}}e^{-i\gamma\frac{B_0}{2\omega}\sin(\omega t)}}$$

Therefore combining these results into our time dependent spinor we get

$$\boxed{\chi(t) = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\gamma\frac{B_0}{2\omega}\sin(\omega t)}\\ e^{-i\gamma\frac{B_0}{2\omega}\sin(\omega t)}\end{pmatrix}}$$

6.3 Solution to Problem 4.33 Part C

The probability of the electron being in the spin-up state is given by the inner product of the spin state it is in and the desired spin state.

$$
\begin{aligned}
P(\uparrow) &= |\langle \chi_-^x | \chi \rangle|^2 \\
&= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right|^2 \\
&= \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma \frac{B_0}{2\omega}\sin(\omega t)} \\ e^{-i\gamma \frac{B_0}{2\omega}\sin(\omega t)} \end{pmatrix} \right|^2 \\
&= \left| \left(\frac{e^{i\gamma \frac{B_0}{2\omega}\sin(\omega t)} - e^{-i\gamma \frac{B_0}{2\omega}\sin(\omega t)}}{2} \right) \right|^2 \\
&= \left| i \sin^2\left(\gamma \tfrac{B_0}{2\omega} \sin(\omega t)\right) \right|^2 \\
&= \boxed{\sin^2\left(\gamma \tfrac{B_0}{2\omega} \sin(\omega t)\right)}
\end{aligned} \tag{5}
$$

6.4 Solution to Problem 4.33 Part C

The electron will be forced to fully flip when the probability of it flipping is 1, $P(\uparrow) = 1$.

$$
1 = P(\uparrow) = \sin^2\left(\gamma \tfrac{B_0}{2\omega} \sin(\omega t)\right) \Longrightarrow \gamma \tfrac{B_0}{2\omega} = \pi/2 \Longrightarrow \boxed{B_0 = \frac{\pi\omega}{\gamma}}
$$

Where we have ignored the sine term in the sine squared term because that is oscillating in time. We just want an arbitrary time when it is maximum of course.