Physics 105 Homework 5

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1 Problem 7.1

1.1 Solution

The period of an equal mass binary star system is found by noting the velocity for uniform circular motion is $v = \frac{2\pi r}{T}$ and using Newtons second law. We know that the particle rotates about its center of mass,r, or one half the distance between the two equal mass stars $\frac{d}{2}$. Hence, the period is

$$\begin{aligned} F =& ma_c \\ G\frac{mm}{d^2} =& m\frac{v^2}{r} \\ G\frac{m}{d^2} =& m\frac{1}{r}\left(\frac{2\pi r}{T}\right)^2 \\ T^2 =& \frac{4\pi^2 d^2 r}{Gm} \\ T =& 2\pi d\sqrt{\frac{r}{Gm}} \\ T =& 2\pi 1.5\times 10^{11}m\sqrt{\frac{7.5\times 10^{10}m}{G\cdot 1.99\times 10^{30}kg}} \\ T =& 22,377,600s = 259\ days \end{aligned} \tag{1}$$

2 Problem 7.2

2.1 Part A

To determine the center of mass we will need to first define a origin, or reference point. The reference point of the sun is convenient because then the suns position is zero and jupiters position is just the radius of jupiters orbit. The center of mass is defined as $\vec{R} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$. Using this definition we find that the center of mass (relative to sun) is just

$$\vec{R} = \frac{m_s\cdot 0 + m_j r_{js}}{m_s + m_j} = \frac{m_j r_{js}}{m_j\left(1+\frac{m_s}{m_j}\right)} = \frac{r_{js}}{\left(1+\frac{m_s}{m_j}\right)} = \frac{5.2AU}{1+1047} = 0.005AU \approx 7.5\times 10^5 km \tag{2}$$

2.2 Part B

The angle that the sun shifts is given by the arc length equation for small arc, or sine function for small angles, where the arc length is the radius of the orbit than the sun makes around the sun-jupiter center of mass, which is just the center of mass position. The radius was found in part A to be $R = 7.5\times 10^5 km$. Hence the angle is

$$\theta = \frac{s}{r} = \frac{R}{r_{js}} = \frac{7.5\times 10^5 km}{1.5\times 10^8 km} = 0.005\ radians = 0.286^\circ \tag{3}$$

3 Problem 7.9

3.1 Part A

The kinetic energy imparted on a molecule by another molecule is given by (assume same type of molecule)

$$\frac{T_2}{T} = \frac{4m_1 m_2}{M^2} sin^2\left(\frac{\theta^*}{2}\right) = \frac{4m^2}{(2m)^2}\frac{1}{2}\left(1 - cos\theta^*\right)$$

In the CM frame the scattering angle is isotropic. This means we can integrate over the entire range of angles in the CM frame and then divide by the range to obtain the average kinetic energy transferred. The range of scattering angles is from 0 to π (in the CM frame). Denote $\frac{T_2}{T}$ by $T'P$. The probability distribution $P(\theta^*)$ is a constant $|1|$ since $\cos\theta^*$ is equiprobable.

$$\begin{aligned}
\delta T' =& \frac{4m^2}{4m^2}\frac{1}{2}\left(1 - cos\theta^*\right) \\
\delta T' P(\theta^*) =& \frac{1}{2}\left(1 - cos\theta^*\right) P(\theta^*) \\
\int^{\overline{T}} \delta T' P(\theta^*) =& \frac{1}{\pi - 0}\int_0^{\pi} \frac{1}{2}\left(1 - cos\theta^*\right) P(\theta^*) d\theta^* \\
\overline{T} =& \frac{1}{2\pi}\left[\theta^* - sin(\theta^*)\right]_0^{\pi} \\
\overline{T} =& \frac{T_2}{T} = \frac{1}{2\pi}\pi = \frac{1}{2}
\end{aligned} \tag{4}$$

This means that every collision the molecule looses half of its previous energy so after n collisions the particles kinetic energy will be given by

$$\frac{T_n}{T} = \overline{T}_n = \frac{1}{2^n}T$$

3.2 Part B

The number of collisions needed to (on average) to reduce the initial velocity by a factor of 1000, or equivalently the kinetic energy by one millionth ($T \propto v^2$), will be given by how many halves square rooted equals one over a thousand. If the particle starts with kinetic energy T it will be reduced by a factor of 1000^2 after n times given by

$$\begin{aligned}
\frac{T}{2^n} =& \frac{T}{1000^2} \\
1000^2 =& 2^n \\
log_2(1000^2) =& n \\
2\frac{log(1000)}{log(2)} =& n = 19.93 \approx 20
\end{aligned} \tag{5}$$

4 Problem 8.1

4.1 Part A

If the rocket needs to reach a maximum height of $50km$ and it is fired directly upward then the initial velocity is found by using elementary kinematics equations, or conservation of energy. We will use energy here, where the initial energy is all kinetic and the final energy is all potential.

$$\frac{mv^2}{2} = mgh \longrightarrow v = \sqrt{2gh} = \sqrt{2(9.8)(50km)} = 0.99\frac{km}{s}$$

4.2 Part B

The initial velocity is $0.99\frac{km}{s}$ and the final velocity is $2\frac{km}{s}$ and the final mass is 100 kg. Now all we don't know from our equation of mass is the initial mass M_0.

$$\begin{aligned}
M =& M_0 e^{-v/u} \\
Me^{v/u} =& M_0 \\
(100kg)e^{0.99/2} =& M_0 = 164kg
\end{aligned} \tag{6}$$

5 Problem 8.3

5.1 Part A

The ratio λ is given as the final mass divided by the initial mass without the mass of the payload. We can use the mass equation (from previous problem) to obtain the initial mass in terms of the payload and initial velocity and ejection velocity. First we write the residual mass as

$$\lambda = \frac{M}{M_0 - m} \longrightarrow M = \lambda M_0 - \lambda m$$

Plugging this value into our standard mass equation (and including the payload) we obtain

$$\begin{aligned}
M = \lambda M_0 - \lambda m + m =& M_0 e^{-v/u} \\
\lambda M_0 - M_0 e^{-v/u} =& m\lambda - m \\
M_0 \left(\lambda - e^{-v/u}\right) =& m(\lambda - 1) \\
M_0 =& m\frac{\lambda - 1}{\lambda - e^{-v/u}} \\
M_0 =& m\frac{1-\lambda}{e^{-v/u}-\lambda}\frac{(-1)}{(-1)} \\
M_0 =& m\frac{1-\lambda}{e^{-v/u}-\lambda}
\end{aligned} \tag{7}$$

5.2 Part B

Let $\lambda = 0.15$ and the ejection velocity $u = 2.5\frac{km}{s}$ we use the initial mass equation with these values and obtain

$$\begin{aligned}
M =& M_0 e^{-v/u} \\
\frac{M}{M_0} =& e^{-v/u} \\
\lambda =& e^{-v/u} \\
ln(\lambda) =& -\frac{v}{u} \\
uln(\lambda) =& -v \\
-2.5\frac{km}{s} ln(0.15) =& v = 4.74\frac{km}{s}
\end{aligned} \tag{8}$$

6 Problem 8.4

6.1 Part A

For a two stage rocket there will be two payloads (or fuel tanks) where the residual mass fraction is given as

$$\lambda = \frac{M}{M_0 - m_1} \longrightarrow M = M_0\lambda - m_1\lambda$$

Plugging this into the mass equation we obtain the equation from the previous problem, but for the second stage since that is what is similar to the previous problem. However, since this initial mass is for the second stage it will be the payload for the first stage.

$$\begin{aligned}
M =& M_{02} e^{-v/u} \\
M_{02}\lambda - m_2\lambda + m_2 =& M_{02} e^{-v/u} \\
M_{02}\lambda - M_{02} e^{-v/u} =& m_2\lambda - m_2 \\
M_{02}\left(\lambda - e^{-v/u}\right) =& m_2(1-\lambda) \\
M_{02} =& \frac{m_2(\lambda - 1)}{\lambda - e^{-v/u}} \\
M_{02} =& \frac{m_2(1-\lambda)}{e^{-v/u} - \lambda}
\end{aligned}$$

Now this mass is the payload mass of the first stage($m = M_{02}$), and we obtain

$$\begin{aligned}
M_0 =& \frac{m(1-\lambda)}{e^{-v/u}-\lambda} \\
M_0 =& \frac{M_{02}(1-\lambda)}{e^{-v/u}-\lambda} \\
M_0 =& \frac{m_2(1-\lambda)}{e^{-v/u}-\lambda}\frac{(1-\lambda)}{e^{-v/u}-\lambda} \\
M_0 =& m_2\left(\frac{1-\lambda}{e^{-v/u}-\lambda}\right)^2
\end{aligned}$$

But we know each the payload is m so we write the equation as

$$M_0 = m\left(\frac{1-\lambda}{e^{-v/u}-\lambda}\right)^2 \tag{9}$$

6.2 Part B

To determine the number of stages to reach escape velocity we look at how fast each stage will increase the velocity. The first stage increases the velocity from 0 to $4.74\frac{km}{s}$. Then second doubles that, the third triples that, etc. To see why this is true we can immediately put our self in a reference frame that is has the rocket at rest, then uses another stage to boost it another $4.74\frac{km}{s}$. Therefore, we can look at the smallest ceiling integer that satisfies

$$[n]v = u = 11.2\frac{km}{s} \longrightarrow [n] = \frac{11.2\frac{km}{s}}{4.74\frac{km}{s}} = [2.363] = 3$$

Hence, it would take at least a 3 stage rocket to reach escape velocity.

6.3 Part C

To find the initial take off mass with a payload of 100kg we just use the equation derived above in part A but with each stage reaching a third of the final velocity. Please note that the squared becomes a cubed because of three stages.

$$\begin{aligned}
M_0 =& m\left(\frac{1-\lambda}{e^{-v/u}-\lambda}\right)^3 \\
M_0 =& 100kg\left(\frac{1-0.15}{e^{-11.2/(3\cdot 2.5)}-0.15}\right)^3 \\
M_0 =& 1.477\times 10^5 kg \approx 1.5\times 10^5 kg
\end{aligned} \tag{10}$$

7 Problem 8.5

7.1 Part A

The earth moves at $28.8\frac{km}{s}$ and the rocket should leave at $38.6\frac{km}{s}$. This means once the rocket escapes it should have a relative velocity of $v = 8.8\frac{km}{s}$. Escaping means that it feels no effects of the gravitational force of earth, hence we can write energy conservation as

$$\begin{aligned}
E_i =& E_f \\
\frac{mv_0^2}{2} - G\frac{Mm}{r_e} =& \frac{mv^2}{2} \\
v_0^2 - 2\frac{GM}{r_e} =& v^2 \\
v_0^2 =& \frac{2GM}{r_e} + v^2 \\
v_0 =& \sqrt{\frac{2GM}{r_e} + v^2} \\
v_0 =& 14.18\frac{km}{s}
\end{aligned} \tag{11}$$

7.2 Part B

The minimum required ejection velocity is right when the denominator of the mass equation goes to zero.

$$\begin{aligned}
e^{-v/3u} - \lambda =& 0 \\
e^{-v/3u} =& \lambda \\
\frac{v}{3u} =& -ln(\lambda) \\
\frac{v}{-3ln(\lambda)} =& u = \frac{14.2\frac{km}{s}}{3ln(10)} = 2.06\frac{km}{s}
\end{aligned} \tag{12}$$

7.3 Part C

We use the equation derived in the previous problem to find the total initial mass

$$\begin{aligned}
M_0 =& m\left(\frac{1-\lambda}{e^{-v/u}-\lambda}\right)^3 \\
M_0 =& 500kg\left(\frac{1-0.1}{e^{-14.18/(3\cdot 2.5)}-0.1}\right)^3 \\
M_0 =& 500kg\left(\frac{0.9}{0.051}\right)^3 \\
M_0 =& 2.752\times 10^6 kg \approx 2.8\times 10^6 kg
\end{aligned} \tag{13}$$

8 Problem 8.9

8.1 Part A

From the figure above we see that the centers of the billiard balls make a equilateral triangle. The force imparted one the two stationary balls is perpendicular to the tangent plane of both contact points. Therefore, the direction of the two velocities of the originally stationary balls are 60° apart. This means they are traveling at 30° and -30° from the original direction of motion. Now we can use our conservation laws to determine the resulting velocities. We start with momentum conservation and noting that there is no net external force applied to the incoming ball in the y-direction, hence no final momentum in the y-direction.

$$\begin{aligned}
&Y-Component \\
P_{iy} =& P_{fy} \\
0 =& mu_1 sin30^\circ + mu_2 sin(-30^\circ) \\
u_1 =& u_2 \longrightarrow |u_1| = |u_2| = u
\end{aligned}$$

$$\begin{aligned}
&X-Component \\
P_{ix} =& P_{fx} \\
mv_0 =& mv + mucos30^\circ + mu30^\circ \\
v_0 - v =& \sqrt{3}u
\end{aligned}$$

Now we can write down energy conservation as our third equation for our three unknowns.

$$\begin{aligned}
E_i =& E_f \\
\frac{mv_0^2}{2} =& \frac{mv^2}{2} + \frac{mu_1^2}{2} + \frac{mu_2^2}{2} \\
v_0^2 =& u_1^2 + u_2^2 + v^2 \\
v_0^2 =& u^2 + u^2 + v^2
\end{aligned}$$

Now we have 3 equations and 3 unknowns. First we will manipulate the energy equation to obtain a difference of squares, and plug in a difference from the momentum equations.

$$\begin{aligned}
v_0^2 - v^2 =& 2u^2\\
(v_0 - v)(v_0 + v) =& 2u^2\\
(v_0 + v)\sqrt{3}u =& 2u^2\\
(v_0 + v)\sqrt{3} =& 2u
\end{aligned}$$

Now plug this back into the x-component momentum equation to obtain

$$\begin{aligned}
v_0 - v =& \sqrt{3}u\\
v_0 - v =& \sqrt{3}(v_0 + v)\frac{\sqrt{3}}{2}\\
v_0 - v =& \frac{3}{2}v_0 + \frac{3}{2}v\\
-\frac{5v}{2} =& \frac{v_0}{2}\\
v =& -\frac{v_0}{5}
\end{aligned}$$

Taking this value and plugging into the previous equation we obtain

$$\begin{aligned}
(v_0 + v)\sqrt{3} =& 2u\\
(v_0 + \frac{-v_0}{5})\sqrt{3} =& 2u\\
\frac{4\sqrt{3}v_0}{5} =& 2u\\
\frac{2\sqrt{3}v_0}{5} =& u
\end{aligned}$$

Now that we know the magnitude of u we can find its vector components by multiplying it by a the sine or cosine of 30 degrees.

$$u_x = \frac{2\sqrt{3}v_0}{5}cos(\pm 30^\circ) = \frac{2\sqrt{3}v_0}{5}\frac{\sqrt{3}}{2} = \frac{3}{5}v_0 \qquad u_y = \frac{2\sqrt{3}v_0}{5}sin(\pm 30^\circ) = \pm\frac{2\sqrt{3}v_0}{5}\frac{1}{2} = \pm\frac{\sqrt{3}}{5}v_0$$

Applying these values to the corresponding particles we find that the resultant velocities are

$$v = \left(-\frac{1}{5}, 0\right)v_0 \qquad u_1 = \left(\frac{\sqrt{3}}{5}, \frac{3}{5}\right)v_0 \qquad u_2 = \left(\frac{\sqrt{3}}{5}, -\frac{3}{5}\right)v_0 \tag{14}$$

9 Extra Credit Problems

9.1 Problem 8.17

We know the gravitational force is an inverse square law, and hence the potential is just inversely proportional to distance between the two objects. Therefore if we define the origin of our coordinate system to be the location of the j-th particle then we can define the potential as the line integral from along any arbitrary path from infinity to the position r_i.

$$V_i = -G\frac{Mm}{r_i} = -r_i \cdot G\frac{Mm}{r_i^2} = -r_i \cdot F_{ij} = r_i \cdot \vec{\nabla}_i V_i$$

Now we evaluate the last expression, where V_i is the potential of the *entire* two body system.

$$
\begin{aligned}
r_i \cdot \vec{\nabla}_i V_i =& \left\langle r_i, \vec{\nabla}_i V_i \right\rangle \\
=& \left\langle (x_i, y_i, z_i), \left(\frac{\partial V_i}{\partial x_i}, \frac{\partial V_i}{\partial y_i}, \frac{\partial V_i}{\partial z_i} \right) \right\rangle \\
=& \left\langle (x_i, y_i, z_i), \left(G\frac{Mm}{x_i^2}, G\frac{Mm}{y_i^2}, G\frac{Mm}{z_i^2} \right) \right\rangle \\
=& x_i \cdot G\frac{Mm}{x_i^2} + y_i \cdot G\frac{Mm}{y_i^2} + z_i \cdot G\frac{Mm}{z_i^2} \\
=& G\frac{Mm}{x_i} + G\frac{Mm}{y_i} + G\frac{Mm}{z_i} \\
=& G\frac{Mm}{r_i} \\
=& - V_i
\end{aligned}
$$

Now we now the potential of each particle pair. Then we can sum the potentials of each to find the total potential V of the system

$$
-V_1 + (-V_2) + (-V_3) + \cdots + (-V_N) = \sum_i r_i \cdot \vec{\nabla} V = -V \tag{15}
$$

Please note this condition expresses the fact that V is a homogeneous function for the coordinates of degree -1.

9.2 Problem 8.18

9.2.1 Part A

Using the conditions of the previous problem we can show that total kinetic and potential energies, T and V satify the virial equation

$$
2T + V = \frac{d^2 K}{dt^2} \qquad\qquad K = \frac{1}{2} \sum_i m_i \vec{r}_i^2
$$

To prove this we just evaluate each side of the expression

$$
\begin{aligned}
2T + V =& 2\frac{1}{2} \sum_i m_i r_i^2 + V \\
=& \sum_i m_i r_i^2 - (-V) \\
=& \sum_i m_i r_i^2 - \sum_i r_i \cdot \vec{\nabla} V \\
=& \sum_i m_i r_i^2 + \sum_i r_i F_i
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\frac{d^2 K}{dt^2} =& \frac{d^2}{dt^2} \frac{1}{2} \sum_i m_i \vec{r}_i^2 \\
=& \frac{d}{dt} \sum_i m_i r_i \cdot \dot{r}_i \\
=& \sum_i m_i \dot{r}_i \cdot \dot{r} + \sum_i r_i m_i \ddot{r}_i \\
=& \sum_i m_i \dot{r}_i^2 + \sum_i r_i F_i
\end{aligned} \tag{17}
$$

9.2.2 Part B

If the system is measured by K, and is neither shrinking nor growing then its derivative is zero. To prove the virial theorem we will use this fact and take the time averages of both sides. We will take the time average from $t = 0$ to $t = \tau$.

$$
\begin{aligned}
2T + V =& \frac{d^2 K}{dt^2} \\
\frac{1}{\tau - 0} \int_0^\tau (2T + V) dt =& \frac{1}{\tau - 0} \int_0^\tau \frac{d^2 K}{dt^2} dt \\
2\frac{1}{\tau} \int_0^\tau T dt + \frac{1}{\tau} \int_0^\tau V dt =& \frac{1}{\tau} \frac{dK}{dt} \Big|_0^\tau \\
2\overline{T} + \overline{V} =& 0 \\
\overline{T} =& -\frac{1}{2} \overline{V}
\end{aligned} \tag{18}
$$

Where $\overline{T}$ denotes the times averaged kinetic energy and $\overline{V}$ denotes the time averaged potential energy. This is equivalent to Chapter 4, Problem 19.