Physics 105 Homework 3

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1 Problem 3.11

A particle of mass m is attached to the end of a light spring of equilibrium length a, whose other end is fixed, so that the spring is free to rotate in a horizontal plane. The tension in the spring is k times its extension. Initially the system is at rest and the particle is given an impulse that starts it moving at right angles to the spring with velocity v. Write down the equations of motion in polar co-ordinates. Given that the maximum radial distance attained is $2a$, use the energy and angular momentum conservation laws to determine the velocity at that point, and to find v in terms of the various parameters of the system. Find also the values of $\ddot{r}$ when $r = a$ and when $r = 2a$.

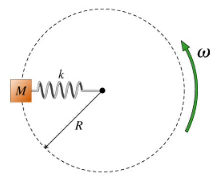


Figure 1: Mass attached to spring rotating.

1.1 Part A

We first write the kinetic energy and potential energy in polar coordinates.

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad V = \frac{1}{2}k\left(r-a\right)^2$$

Now by minimizing the Lagrangian we can determine the equations of motion using the Euler-Lagrange Equations with $L = T - V$.

$$\frac{\partial L}{d\dot{r}} = m\dot{r} \qquad \frac{\partial L}{dr} = mr\dot{\theta}^2 - k(r-a)$$

$$\frac{\partial L}{d\dot{\theta}} = r^2\dot{\theta} \qquad \frac{\partial L}{d\theta} = 0$$

Putting them together we obtain

$$\frac{d}{dt}\frac{\partial L}{d\dot{r}} - \frac{\partial L}{dr} = \frac{d}{dt}\left(m\dot{r}\right) - mr^2\dot{\theta} + k(r-a) = 0$$

$$\frac{d}{dt}\frac{\partial L}{d\dot{\theta}} - \frac{\partial L}{d\theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) - 0 = 0$$

From here it is easy to see that

$$m\ddot{r} = mr\dot{\theta}^2 - k\left(r-a\right) \quad (1) \qquad mr^2\dot{\theta} = J = constant \quad (2)$$

This tells us that the only acceleration felt by the mass is the radial, or centripetal acceleration, which means it is experiencing uniform circular motion.

1.2 Part B

To find the velocity at $r = 2a$ to we will apply conservation of angular momentum. At $t = 0$ the angular momentum is mva and when the mass reaches $r = 2a$ its angular momentum is $mv_{2a}(2a)$. We see that the velocity at $2a$ is

$$mva = mv_{2a}(2a) \longrightarrow v_{2a} = \frac{v}{2} \quad (3)$$

1.3 Part C

We know that energy is constant throughout the oscillation. There is no external forces acting acting so the initial energy is equal to the final energy (at $r = 2a$), where $v_{2a} = \frac{v}{2}$.

$$\begin{aligned} E_a =& E_{2a} \\ \frac{mv^2}{2} =& \frac{mv_{2a}^2}{2} + \frac{k}{2}(r-a)^2 \\ v^2 =& \frac{v^2}{4} + \frac{k}{m}a^2 \\ v^2\left(1-\frac{1}{4}\right) =& \frac{k}{m}a^2 \\ v^2 =& \frac{4k}{3m}a^2 \end{aligned} \tag{4}$$

1.4 Part D

To find the radial acceleration at $r = a$ we just use our equations of motion, with $\dot{\theta} = \frac{v}{r}$.

$$\begin{aligned} m\ddot{r} =& mr\dot{\theta}^2 - k(r-a) \\ m\ddot{r} =& mr\left(\frac{v}{r}\right)^2 - k(r-a) \\ m\ddot{r} =& ma\left(\frac{\frac{4k}{3m}a^2}{a^2} - 0\right) \\ \ddot{r} =& \frac{4ka}{3m} \end{aligned} \tag{5}$$

1.5 Part E

To find the radial acceleration at $r = 2a$ we just use our equations of motion, with $\dot{\theta} = \frac{v}{r}$.

$$\begin{aligned} m\ddot{r} =& mr\dot{\theta}^2 - k(r-a) \\ m\ddot{r} =& mr\left(\frac{v}{r}\right)^2 - k(r-a) \\ m\ddot{r} =& 2ma\left(\frac{\frac{4k}{3m}a^2}{4a^2} - ka\right) \\ \ddot{r} =& -\frac{k}{m}a + \frac{ka}{6m} \\ \ddot{r} =& -\frac{5k}{6m}a \end{aligned} \tag{6}$$

2 Problem 3.14

A wedge-shaped block of mass M rests on a smooth horizontal table. A small block of mass m is placed on its upper face, which is also smooth and inclined at an angle α to the horizontal. The system is released from rest. Write down the horizontal component of momentum, and the kinetic energy of the system, in terms of the velocity v of the wedge and the velocity u of the small block relative to it. Using conservation of momentum and the equation for the rate of change of kinetic energy, find the accelerations of the blocks. Given that $M = 1kg$ and $m = .250kg$, find the angle that will maximize the acceleration of the wedge.

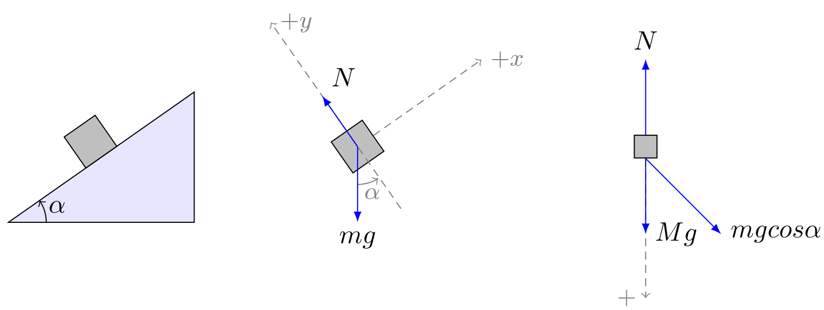


Figure 2: A wedge mass and a block on it. No friction in the system. With free body diagrams of both. Note: The right free body diagram is the wedge, just drawn as a generic square.

2.1 Part A

The horizontal component of momentum can be written easily since initially it has no momentum.

$$P_{xi} = P_{xf} = 0 = m\vec{u} + M\vec{v} \tag{7}$$

To write the kinetic energy of the system we must first investigate the relative motions of the system. The box slides down the wedge with velocity u, but the wedge is also sliding away from the box with velocity v. Hence the relative velocity of the box with respect to the ground is $u + v$. The wedge has velocity v relative to the ground. The kinetic energy of the system is

$$\begin{aligned} T =& \frac{1}{2}m(\vec{u}+\vec{v})^2 + \frac{1}{2}M\vec{v}^2 \\ =& \frac{1}{2}m\vec{u}\cdot\vec{u} + \frac{1}{2}m\vec{v}\cdot\vec{v} + m\vec{u}\cdot\vec{v} + \frac{1}{2}M\vec{v}\cdot\vec{v} \\ =& \frac{1}{2}mu^2 + \frac{1}{2}mv^2 + muvcos\alpha + \frac{1}{2}Mv^2 \end{aligned} \tag{8}$$

2.2 Part B

To determine the accelerations of the two bodies we will use the Hamiltonian approach by extremizing the Lagrangian. First we will need to calculate the potential energy function. The wedge has no potential energy, but the box does which is

$$V(x_1) = mgx_1sin(\alpha)$$

Now we just need to write down the Euler-Lagrange equations and solve them. The Lagrangian is

$$L = T - V = \frac{1}{2}mu^2 + \frac{1}{2}mv^2 + muvcos\alpha + \frac{1}{2}Mv^2 - mgx_1sin(\alpha)$$

The respective derivatives are

$$\frac{\partial L}{du} = mu + mvcos\alpha \qquad \frac{\partial L}{dx_1} = mgsin\alpha$$

$$\frac{\partial L}{dv} = mv + mucos\alpha + Mv \qquad \frac{\partial L}{dx_2} = 0$$

Inserting into the Euler-Lagrangian equations we obtain

$$\frac{d}{dt}\frac{\partial L}{du} - \frac{\partial L}{dx_1} = \frac{d}{dt}(mu + mvcos\alpha) - mgsin\alpha = m\dot{u} + m\dot{v}cos\alpha - mgsin\alpha = 0$$

$$\frac{d}{dt}\frac{\partial L}{dv} - \frac{\partial L}{dx_2} = \frac{d}{dt}(mv + mucos\alpha + Mv) - 0 = m\dot{v} + m\dot{u}cos\alpha + M\dot{v} = 0 \longrightarrow (1+\tfrac{M}{m})\dot{v} = -\dot{u}cos\alpha$$

From here we can solve for the individual accelerations simultaneously, since we have two equations and two unknowns.

$$\begin{aligned} m\dot{u} + m\dot{v}cos\alpha =& mgsin\alpha \\ m\dot{u} + m\frac{-\dot{u}cos\alpha}{(1+\frac{M}{m})}cos\alpha =& mgsin\alpha \\ m\dot{u}(1+\tfrac{M}{m}) - m\dot{u}cos^2\alpha =& (1+\tfrac{M}{m})mgsin\alpha \\ \dot{u}\left(\tfrac{M}{m} + 1 - cos^2\alpha\right) =& (1+\tfrac{M}{m})gsin\alpha \\ \dot{u} =& \frac{(1+\frac{M}{m})gsin\alpha}{\left(\frac{M}{m}+1-cos^2\alpha\right)} \\ \dot{u} =& \frac{\frac{1}{m}(M+m)gsin\alpha}{\frac{1}{m}(M+msin^2\alpha)} \\ \dot{u} =& \frac{(M+m)gsin\alpha}{(M+msin^2\alpha)} \end{aligned} \tag{9}$$

$$\begin{aligned} (1+\tfrac{M}{m})\dot{v} =& -\frac{(M+m)gsin\alpha}{(M+msin^2\alpha)}cos\alpha \\ \frac{1}{m}(M+m)\dot{v} =& -\frac{(M+m)gsin\alpha}{(M+msin^2\alpha)}cos\alpha \\ \dot{v} =& -\frac{mgsin\alpha cos\alpha}{(M+msin^2\alpha)} \end{aligned} \tag{10}$$

2.3 Part C

To find the angle that will maximize the acceleration of the wedge we need find the derivative of the acceleration function with respect to α, and set it equal to zero. Computing the derivative with Matlab I found that the angle that will give maximal wedge acceleration is given by

$$\frac{d}{d\alpha}\ddot{v} = \frac{2mg((2M+m)cos(2\alpha) - m)}{(2M - mcos(2\alpha) + M^2)} \longrightarrow (2M+m)cos(2\alpha) = m$$

Where we have gotten rid of the denominator because it can not make the derivative zero. The angle is then just

$$cos(2\alpha) = \frac{m}{2M+m} \longrightarrow \alpha = \frac{cos^{-1}(\frac{0.250}{2.125})}{2} = 41.8° \tag{11}$$

Plotting the wedge acceleration vs the angle of incline I have confirmed that around 40 degrees the maximum occurs as seen in the figure below

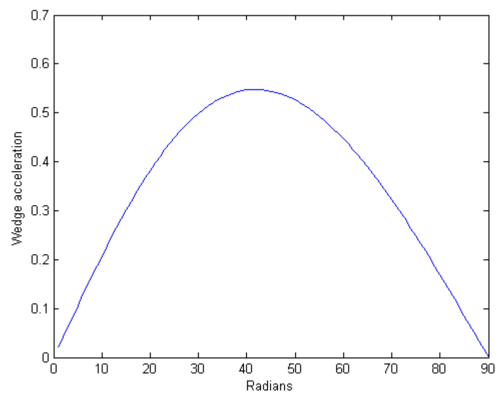


Figure 3: A graph generated using Matlab to observe wedge acceleration vs α.

3 Problem 3.15

A particle starts from rest and slides down a smooth curve under gravity. Find the shape of the curve that will minimize the time taken between two given points

3.1 Solution

To minimize the time we can use calculus of variations techniques. First we define an arc length in differential form

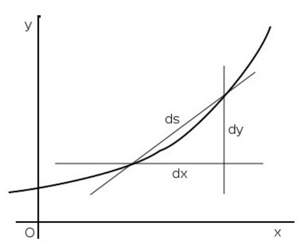


Figure 4: Infinitesimal arc length of an arbitrary path.

We know that the initial energy equals the final energy since there are no non-conservative forces. We see that the kinetic energy at any point in time is equal to the potential energy at that point relative to the origin. Hence,

$$\frac{1}{2}mv^2 = mgz \longrightarrow v = \sqrt{2gz}$$

Now we need to use our definition of velocity. If we choose an "axis" of our coordinate system to be the path of the ball we can just use the arc length definition, $ds = \sqrt{dx^2 + dy^2}$. With this our velocity is defined as

$$v = \frac{ds}{dt}\sqrt{2gz} \longrightarrow \int^t dt = \int^s \frac{ds}{\sqrt{2gz}}$$

Now we integrate the time functional dt. Note that I have been unconventional and defined $\dot{z} = \frac{dz}{dx} \neq \frac{dz}{dt}$.

$$\begin{aligned}\int^t &= \int^s \frac{ds}{\sqrt{2gz}} \\ &= \int^s \frac{\sqrt{dx^2 + dz^2}}{\sqrt{2gz}} \\ &= \int^z \frac{\sqrt{1 + \left(\frac{dx}{dz}\right)^2}dx}{\sqrt{2gz}}\end{aligned} \tag{12}$$

Now we extremize the functional using Euler-Lagrange technique to obtain

$$\frac{\partial F}{d\dot{x}} = \frac{\dot{x}}{\sqrt{z}\sqrt{1+\dot{x}^2}} \qquad\qquad \frac{\partial F}{dx} = 0$$

Using the Euler-Lagrange Equation we find that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial F}{d\dot{x}}\right) - \frac{\partial F}{dx} &=0 \\ \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{2gz}\sqrt{1+\dot{x}^2}}\right) &=0 \\ \frac{\dot{x}}{\sqrt{2gz}\sqrt{1+\dot{x}^2}} &=C \\ \dot{x} &=\sqrt{2gz}\sqrt{1+\dot{x}^2}C \\ \dot{x}^2 &=2gzC^2 + 2gzC^2\dot{x}^2 \\ \dot{x}^2\left(1 - 2gzC^2\right) &=2gzC^2 \\ \frac{dx}{dz} &=\frac{\sqrt{2gz}C}{\sqrt{1-2gzC^2}} \\ \int^x dx &= \int_{z_1}^{z_2} \frac{\sqrt{2gz}C}{\sqrt{1-2gzC^2}}dz\end{aligned} \tag{13}$$

If we let $2gzC^2 = sin^2(\frac{\theta}{2}) = \frac{1}{2}(1 - cos\theta)$ then $2gC^2dz = 2cos\frac{\theta}{2}sin\frac{\theta}{2}d\theta$ we obtain

$$\begin{aligned}x &= \int^{\theta(z)} \frac{1}{gC^2}\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}cos\frac{\theta}{2}sin\frac{\theta}{2}d\theta \\ x &= \int^{\theta(z)} \frac{1}{2gC^2}\left(1 - cos\theta\right)d\theta \\ x &=\frac{1}{2gC^2}\left(\theta - sin\theta\right)\end{aligned} \tag{14}$$

Where the z-direction parametric equation is obtained by the previous substitution

$$z = \frac{1}{2gC^2}\left(1 - cos\theta\right) = \frac{1}{gC^2}sin^2\frac{\theta}{2} \tag{15}$$

4 Problem 3.18

Parabolic coordinates (ξ, η) in a plane are defined by $\xi = r + x$, $\eta = r - x$. Find x and y in terms of xi and η. Find its kinetic energy and its equation of motion.

4.1 Part A

To find the coordinates of of the particle in terms of x and y we solve for each in terms of the parabolic coordinates. We know that r is defined as $r = \sqrt{x^2 + y^2}$. First lets look at what parabolic coordinates look like

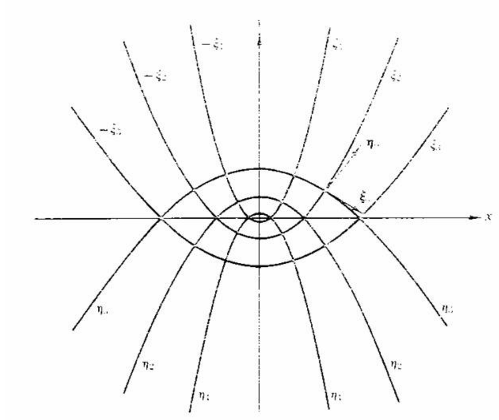


Figure 5: Generic parabolic coordinate axes.

Using the previous information we can solve for x and y.

Solve for x by solving for r and equating.

$$\begin{aligned} \xi - x =& \eta + x \\ 2x =& \xi - \eta \\ x =& \frac{\xi - \eta}{2} \end{aligned} \tag{16}$$

Solve for y by using the definition of r.

$$\begin{aligned} \sqrt{x^2 + y^2} =& r \\ \sqrt{x^2 + y^2} =& \eta + x \\ x^2 + y^2 =& \eta^2 + 2\eta x + x^2 \\ y^2 =& \eta^2 + 2\eta \frac{\xi - \eta}{2} \\ y^2 =& \eta\xi \\ y =& \sqrt{\xi\eta} \end{aligned} \tag{17}$$

4.2 Part B

We know that the kinetic energy of a particle is given by

$$T = \frac{1}{2} m \dot{\vec{x}}^2 + \frac{1}{2} m \dot{\vec{y}}^2$$

Now we must find the derivatives of the positions and square them. The respective derivatives are

$$\begin{aligned} \dot{x} =& \frac{\dot{\eta}}{2} + \frac{\dot{\xi}}{2} \\ \dot{x}^2 =& \frac{1}{4}\left(\dot{\eta}^2 + 2\dot{\eta}\dot{\xi} + \dot{\xi}^2\right) \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{y} =& \frac{\sqrt{\eta}}{2\sqrt{\xi}}\dot{\xi} + \frac{\sqrt{\eta}}{2\sqrt{\xi}}\dot{\eta} \\ \dot{y}^2 =& \left(\frac{\sqrt{\eta}}{2\sqrt{\xi}}\dot{\xi} + \frac{\sqrt{\eta}}{2\sqrt{\xi}}\dot{\eta}\right)^2 \\ \dot{y}^2 =& \frac{\eta}{4\xi}\dot{\xi} + \frac{\dot{\xi}\dot{\eta}}{2} + \frac{\xi}{4\eta}\dot{\eta} \end{aligned} \tag{19}$$

Now plugging these into the kinetic energy equation we find that

$$\begin{aligned} T =& \frac{1}{2} m \dot{\vec{x}}^2 + \frac{1}{2} m \dot{\vec{y}}^2 \\ T =& \frac{1}{2} m \left(\dot{\vec{x}}^2 + \dot{\vec{y}}^2\right) \\ T =& \frac{1}{2} m \left(\frac{1}{4}\left(\dot{\eta}^2 + 2\dot{\eta}\dot{\xi} + \dot{\xi}^2\right) + \frac{\eta}{4\xi}\dot{\xi} + \frac{\dot{\xi}\dot{\eta}}{2} + \frac{\xi}{4\eta}\dot{\eta}\right) \\ T =& \frac{1}{8} m \left(\dot{\eta}^2 + 4\dot{\eta}\dot{\xi} + \dot{\eta}^2 + \frac{\xi}{\eta}\dot{\eta} + \frac{\eta}{\xi}\dot{\xi}\right) \\ T =& \frac{m}{8}(\eta + \xi)\left(\frac{\dot{\eta}^2}{\eta} + \frac{\dot{\xi}^2}{\xi}\right) \end{aligned} \tag{20}$$

4.3 Part C

To find the equations of motion we will use the Euler-Lagrange equations. The Lagrangian is

$$L = T - V = \frac{m}{8}(\eta + \xi)\left(\frac{\dot{\eta}^2}{\eta} + \frac{\dot{\xi}^2}{\xi}\right) - V(\eta, \xi)$$

Now we extremize this functional. We also use the relation $F = -\frac{\partial V}{\partial r}$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}} - \frac{\partial L}{\partial \xi} &= 0 \\
\frac{m\ddot{\xi}}{4} + \frac{m\eta}{8\xi} + \frac{m\dot{\xi}\eta}{8\xi^2} - \frac{m\dot{\eta}}{8\eta} - \frac{\partial V}{\partial \xi} &= 0 \\
\frac{m}{4}\left(-\ddot{\xi} - \frac{\dot{\xi}\eta}{2\xi^2} + \frac{\dot{\eta}\xi}{2\xi^2} - \frac{\dot{\xi}\eta}{2\eta^2} + \frac{\dot{\eta}}{2\eta}\right) &= F_\xi \\
\frac{m}{4}\left(-\ddot{\xi} - \frac{\dot{\xi}\eta}{2\xi^2} + \frac{\dot{\eta}\xi}{2\xi^2} - \frac{\dot{\xi}\eta}{2\eta^2} + \frac{\dot{\eta}}{2\eta}\right) &= F_\xi
\end{aligned}$$

$$\frac{m}{4}\left[\frac{\eta + \xi}{\xi}\ddot{\xi} - \frac{1}{2}\eta\left(\frac{\dot{\xi}}{\xi} - \frac{\dot{\eta}}{\eta}\right)^2\right] = F_\xi$$

The equations of motion for the parameter η are found similarly. If we note the symmetry of the our Lagrangian we can see that the equation of motion should be look the same just with ξ and η switched.

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}} - \frac{\partial L}{\partial \eta} &= 0 \\
\frac{m\ddot{\eta}}{4} + \frac{m\xi}{8\eta} + \frac{m\dot{\eta}\xi}{8\eta^2} - \frac{m\dot{\xi}}{8\xi} - \frac{\partial V}{\partial \eta} &= 0 \\
\frac{m}{4}\left(-\ddot{\eta} - \frac{\dot{\eta}\xi}{2\eta^2} + \frac{\dot{\xi}\eta}{2\eta^2} - \frac{\dot{\eta}\xi}{2\xi^2} + \frac{\dot{\xi}}{2\xi}\right) &= F_\eta \\
\frac{m}{4}\left(-\ddot{\xi} - \frac{\dot{\xi}\eta}{2\xi^2} + \frac{\dot{\eta}\xi}{2\xi^2} - \frac{\dot{\xi}\eta}{2\eta^2} + \frac{\dot{\eta}}{2\eta}\right) &= F_\eta \\
\frac{m}{4}\left[\frac{\xi + \eta}{\eta}\ddot{\eta} - \frac{1}{2}\xi\left(\frac{\dot{\eta}}{\eta} - \frac{\dot{\xi}}{\xi}\right)^2\right] &= F_\eta
\end{aligned}$$

Where the last step factored using Matlab factoran function.

5 Problem 3.19

Write down the equations of motion in polar co-ordinates for a particle of unit mass moving in a plane under a force with potential energy function $V = -kln r + cr + grcos\theta$, where k, c and g are positive constants. Find the positions of equilibrium (a) if $c > g$, and (b) if $c < g$. By considering the equations of motion near these points, determine whether the equilibrium is stable.

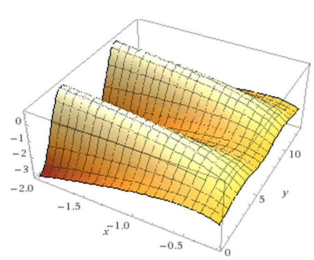


Figure 6: A plot of the potential. Generated by Wolfram Alpha.

5.1 Part A

The equations of motion in polar coordinates are given by

$$F_r = -\frac{\partial V}{\partial r} = \frac{k}{r} - c - gcos\theta \tag{21}$$

$$F_\theta = -\frac{\partial V}{\partial \theta} = grsin\theta \tag{22}$$

$$F_z = -\frac{\partial V}{\partial z} = 0 \tag{23}$$

5.2 Part B

To find the positions of equilibrium we need to find where the sum of the forces are equal to zero. For the angular part we find that

$$0 = grsin\theta \longrightarrow \theta = n\pi \qquad \text{for n} = 0, 1, 2, 3...$$

Now we solve for the radial force when it is equal to zero.

$$0 = \frac{k}{r} - c - gcos\theta \longrightarrow r = \frac{k}{c + gcos\theta} = \frac{k}{c \pm g}$$

To determine if the equilibrium points are stable we take the derivative of the force to observe if the potential function at that point is concave up or concave down.

5.2.1 $c < g$

$$F_r' = -\frac{\partial^2 V}{\partial r^2} = -\frac{k}{r^2} < 0 \longrightarrow \text{Unstable}$$

We see that the radial function is unstable regardless of c or g.

$$F_\theta' = -\frac{\partial^2}{\partial \theta^2} = grcos\theta \longrightarrow Concave\ Down\ for\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

However, it is only in equilibrium at $\theta = n\pi$. **So it is in unstable equilibrium at** $\theta = 0$, and **in stable equilibrium at** $\theta = \pi$.

5.2.2 $g < c$

$$F_r' = -\frac{\partial^2}{\partial r^2} = -\frac{k}{r^2} < 0 \longrightarrow \text{Unstable}$$

We see that the radial function is unstable regardless of c or g.

$$F_\phi' = -\frac{\partial^2}{\partial \theta^2} = grcos\theta \longrightarrow Stable\ for\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

We see that **it is also unstable at** $\theta = 0$ for $g < c$.

6 Problem 4.3

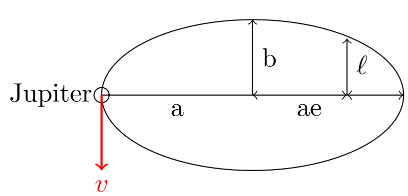


Figure 7: The elliptical orbit of Jupiter

6.0.3 Part A

By using the relation (3.26 from the textbook) $\frac{dA}{dt} = \frac{J}{2m}$ we can take this derivative relation to non-infinitesimal quantities to obtain $\frac{\Delta A}{\tau} = \frac{J}{2m}$ where the area of an ellipse is $A = \pi ab$. Referring to the figure above we find that the length ℓ is defined as $\ell = \frac{J^2}{|k|m}$, and $a = \frac{|k|}{2|E|}$ and $b^2 = a\ell = \frac{J^2}{2m|E|}$. Using all this we can solve for the orbital period τ. We will also solve for τ with equation of period for oscillator given that Jupiter's potential energy function is expressed as $V(r) = -G\frac{M_{sun}m_{jup}}{r}$.

$$\begin{aligned}
\frac{J}{2m} &= \frac{\Delta A}{\tau} \\
\frac{J^2}{4m^2} &= \frac{\Delta A^2}{\tau^2} \\
\tau^2 m|k|\ell &= 4\pi^2 m^2 a^2 b^2 \\
\tau^2 &= \frac{4\pi^2 m^2 a^2 b^2}{m|k|\ell} \\
\tau^2 &= \frac{4\pi^2 m^2 a^3 \ell}{m|k|\ell} \\
\tau^2 &= \frac{4\pi^2 m a^3 \ell}{|k|} \\
\tau &= 2\pi\sqrt{\frac{m}{|k|}a^3} \\
\tau &= 2\pi\sqrt{\frac{(5.20 \cdot 1.5\times 10^{11}m)^3}{G\cdot 1.99\times 10^{30}kg}} \\
\tau &= 375,692,320s \longrightarrow 11.92 \quad \text{years}
\end{aligned}$$

First we will calculate the second derivative of the potential

$$V''(x) = \frac{GM_{sun}m_{jup}}{a^3}$$

Then use the harmonic oscillator period equation

$$\begin{aligned}
\tau &= 2\pi\sqrt{\frac{m}{k}} \\
&= 2\pi\sqrt{\frac{m_{jup}}{\frac{GM_{sun}m_{jup}}{a^3}}} \\
&= 2\pi\sqrt{\frac{a^3}{GM_{sun}}} \\
&= 375,692,320s \longrightarrow 11.92 \quad \text{years}
\end{aligned} \tag{24}$$

We see that the orbit can be modeled as an harmonic oscillator as long they are in a central potential, regardless of orbit shape.

6.1 Part B

To determine the orbital speed we can make an assumption that Jupiter's orbit is circular (its very close to circular). Then all we need to do is divide its total distance by the total time it takes for one orbit. Doing this we find that

$$\overline{v} = \frac{2\pi\overline{R}}{\tau} = \frac{2\pi \cdot 5.2 \cdot 1.5\times 10^8 km}{375,692,320s} = 13.06\frac{km}{s} \tag{25}$$

7 Problem 4.4

The orbit of an asteroid extends from the Earths to Jupiters, just touching both. Find its orbital period.

7.1 Solution

To find the orbital period we just use Kepler's Law (Derived in the previous problem). Since we know the semi major axis of both jupiter and earth, we can can average them to find the semi major axis of the asteroid, $a_a = \frac{a_j + a_e}{2}$. Plugging this into Kepler's Law we find that

$$\tau = 2\pi\sqrt{\frac{a_a^3}{GM_s}} = 2\pi\sqrt{\frac{\left(\frac{a_j+a_e}{2}\right)^3}{GM_s}} = 2\pi\sqrt{\frac{(4.65\times 10^{11}m)^3}{G\cdot 1.99\times 10^{30}kg}} = 5.47 \; years \tag{26}$$

An interesting point: The time is also just the period of earth subtracted from the period of jupiter and divided by two.

$$\begin{aligned}
\tau &= 2\pi\sqrt{\frac{(5.2\cdot 1.5\times 10^{11}m)^3}{GM_{sun}}} \\
\tau &= 2\pi\sqrt{\frac{(5.2\cdot 1.5\times 10^{11}m)^3}{G\cdot 1.99\times 10^{30}kg}} \\
&= 375,692,320s \longrightarrow 11.92 \;\; \text{years}
\end{aligned}$$

$$\begin{aligned}
\tau &= 2\pi\sqrt{\frac{(1.5\times 10^{11}m)^3}{GM_{sun}}} \\
\tau &= 2\pi\sqrt{\frac{(1.5\times 10^{11}m)^3}{G\cdot 1.99\times 10^{30}kg}} \\
&= 31,536,000s \longrightarrow 1 \;\; \text{year}
\end{aligned}$$

Subtracting Jupiter's orbital period from Earth's orbital period and dividing by two we find that the orbital period of the asteroid as

$$\frac{\tau_{jup} - \tau_{earth}}{2} = \frac{11.92 - 1}{2} = \frac{10.92yr}{2} = 5.46 \;\; \text{year}$$

8 Problem 4.8

The Sun has an orbital speed of about $220\frac{km}{s}$ around the centre of the Galaxy, whose distance is 28,000 light years. Estimate the total mass of the Galaxy in solar masses.

8.1 Solution

To estimate the mass of the galaxy we will assume that all the mass that has a gravitational effect on the sun is within the suns orbit. We will also treat all the mass inside the suns orbit is at the center of the galaxy. Now we just use Newtons law and solve for the galaxies mass.

$$\begin{aligned}\sum F =& ma_c \\ G\frac{M_{gal}M_{sun}}{R^2} =& M_{sun}\frac{v^2}{R} \\ \frac{M_{gal}}{M_{sun}} =& \frac{v^2 R}{M_{sun}G} \\ =& \frac{\left(220\times 10^3 \frac{m}{s}\right)^2 \cdot 2.65\times 10^{20}}{G\cdot 1.99\times 10^{30} kg} \\ =& 9.85\times 10^{10} \approx 1\times 10^{11} \text{ Solar Masses}\end{aligned} \tag{27}$$

9 Problem 4.9

A particle of mass m moves under the action of a harmonic oscillator force with potential energy $\frac{kr^2}{2}$. Initially, it is moving in a circle of radius a. Find the orbital speed v. It is then given a blow of impulse mv in a direction making an angle α with its original velocity. Use the conservation laws to determine the minimum and maximum distances from the origin during the subsequent motion. Explain your results physically for the two limiting cases $\alpha = 0$ and $\alpha = \pi$.

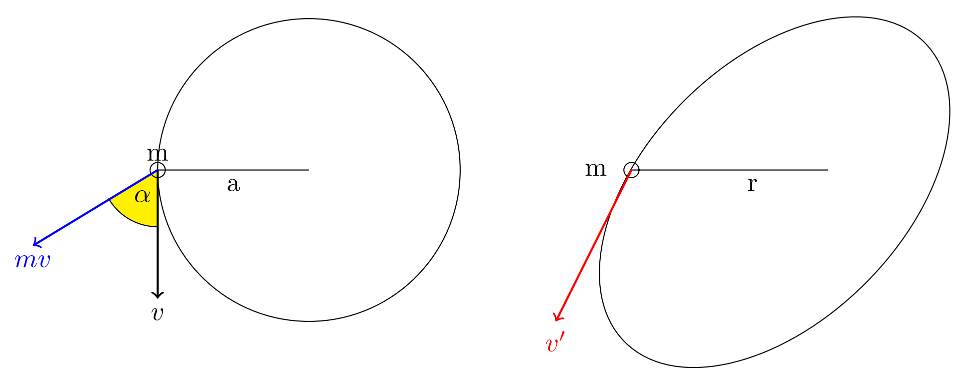


Figure 8: A particle in circular motion. Then it is given an impulse of mv. After it will have an elliptical orbit

9.1 Part A

To find the orbital velocity we will use newtons law. Since the potential is $\frac{kr^2}{2}$ we just differentiate with respect to r to get the force, hence, $F = kr$. Now equating the force on the particle to the its mass times its acceleration (only centripetal), at $r = a$, we obtain

$$ka = \frac{mv^2}{a} \longrightarrow v = a\sqrt{\frac{k}{m}} \tag{28}$$

9.2 Part B

To find the resulting apsis and periapsis due to the impulse we will use conservation of angular momentum from immediately after the impulse to where its apsis and periapsis are. We know that at these two points the angle between the position vector and the velocity vector are perpendicular hence $L = mrvsin(90°) = mrv$.

Start with angular momentum conservation.

$$\begin{aligned} L_i =& L_f \\ mva + mvacos\alpha =& mrv_f \\ va(1 + cos\alpha) =& rv_f \\ v_f =& \frac{va(1 + cos\alpha)}{r} \\ v_f^2 =& \frac{v^2a^2(1 + cos\alpha)^2}{r^2} \end{aligned}$$

Now plug this value into our conservation of energy equation.

$$\begin{aligned} E_i =& E_f \\ \frac{m(2v)^2}{2} + \frac{ka^2}{2} =& \frac{mv_f^2}{2} + \frac{kr^2}{2} \\ 4mv^2 + ka^2 =& mv_f^2 + kr^2 \\ 4mv^2 + ka^2 =& m\frac{v^2a^2(1 + cos\alpha)^2}{r^2} + kr^2 \\ (4mv^2 + ka^2)r^2 =& mv^2a^2(1 + cos\alpha)^2 + kr^4 \\ kr^4 - (4mv^2 + ka^2)r^2 + & mv^2a^2(1 + cos\alpha)^2 = 0 \end{aligned}$$

If we make the substitution $y = r^2$ we have a quadratic equation that can easily be solved by hand.

$$\begin{aligned} kr^4 - (4mv^2 + ka^2)yr^2 + mv^2a^2(1 + cos\alpha)^2 =& 0 \\ ky^2 - (4mv^2 + ka^2)y + mv^2a^2(1 + cos\alpha)^2 =& 0 \\ y =& \frac{-(4mv^2 + ka^2) \pm \sqrt{(4mv^2 + ka^2)^2 - 4kmv^2a^2(1 + cos\alpha)^2}}{2k} \\ r^2 =& \frac{-(4mv^2 + ka^2) \pm \sqrt{(4mv^2 + ka^2)^2 - 4kmv^2a^2(1 + cos\alpha)^2}}{2k} \end{aligned}$$

By using Wolfram Alpha I was able to show that this solution was equivalent to the answer in the book, if we used the relationship $v = a\sqrt{\frac{k}{m}}$. The result from simplifying the quadratic is

$$r^2 = \frac{a^2}{2}\left(3 + 2cos\alpha \pm \sqrt{5 + 4cos\alpha}\right) \tag{29}$$

9.3 Part C

We will now find the apsis and periapsis of the resultant orbit in the limiting cases.

9.3.1 Limiting Case 1: $\alpha = 0$

If $\alpha = 0$ that means the impulse was directed in the direction of motion of the particle. This means it will go out farther, slow down, then come back in close to the focus. Mathematically, we plug in $\alpha = 0$ to the previous equation

$$\begin{aligned} r^2 =& \frac{a^2}{2}\left(3 + 2cos(0) \pm \sqrt{5 + 4cos(0)}\right) \\ r^2 =& \frac{a^2}{2}(5 \pm 3) \\ r^2 =& a^2 \quad \text{or} \quad 4\text{a}^2 \\ r =& a \quad \text{or} \quad 2\text{a} \end{aligned} \tag{30}$$

Now we see that the particle it would double its distance from the origin then return to its original distance to the origin.

9.3.2 Limiting Case 1: $\alpha = \pi$

If $\alpha = \pi$ that means the impulse was directed in the opposite direction of motion of the particle. This means it will stop completely then fall to the focus. Mathematically, we plug in $\alpha = \pi$ to the previous equation

$$\begin{aligned} r^2 =& \frac{a^2}{2}\left(3 + 2cos(\pi) \pm \sqrt{5 + 4cos(\pi)}\right) \\ r^2 =& \frac{a^2}{2}(1 \pm 1) \\ r^2 =& a^2 \quad \text{or} \quad 0 \\ r =& a \quad \text{or} \quad 0 \end{aligned} \tag{31}$$

Here we see that in fact the particle would fall down toward the focus, and if the focus had any dimension there would be a collision and the orbit would stop (or be disturbed). Mathematically it could just orbit from 0 to the original distance from the origin.

10 Problem 4.10

Write down the effective potential energy function $U(r)$ for the system described in Chapter 3, Problem 11. Initially, the particle is moving in a circular orbit of radius $2a$. Find the orbital angular velocity ω in terms of the natural angular frequency ω_0 of the oscillator when not rotating. If the motion is lightly disturbed, the particle will execute small oscillations about the circular orbit. By considering the effective potential energy function $U(r)$ near its minimum, find the angular frequency ω_1 of small oscillations. Hence describe the disturbed orbit qualitatively.

10.1 Part A

We can write the energy in polar coordinates

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

Where we know the potential is purely radial since the force is purely radial. We can use the definition of angular momentum $J = mr^2\dot{\theta}$, and the fact that this quantity is conserved since the torque is zero (force is radial), to rewrite the energy equation

$$E = \frac{m}{2}\dot{r}^2 + \frac{mr^2J^2}{2m^2r^4} + \int_a^r k(r-a) = \frac{m}{2}\dot{r}^2 + U(r)$$

Where we have defined

$$U(r) = \frac{J^2}{2mr^2} + \frac{k}{2}(r-a)^2\Big|_a^r = \frac{J^2}{2mr^2} + \frac{k}{2}(r-a)^2 \tag{32}$$

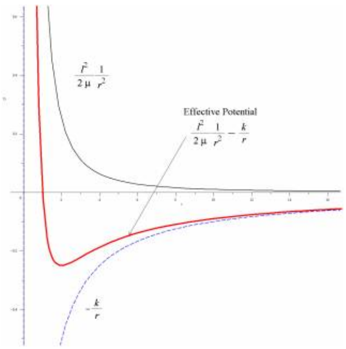


Figure 9: A generic plot of a central potential.

10.2 Part B

If the particle is moving in a circle the radial force will give it the centripetal acceleration needed to stay in the circle. The centripetal acceleration is given by $a_c = \omega^2 r$. We also note that by definition $\omega_0 = \sqrt{\frac{k}{m}}$.

$$\begin{aligned} F =& ma_c \\ k(2a-a) =& m\omega^2(2a) \\ ka =& 2m\omega^2 a \\ \sqrt{\frac{k}{2m}} =& \omega \\ \omega =& \frac{\omega_0}{\sqrt{2}} \end{aligned} \tag{33}$$

10.3 Part C

If the mass is given an radial impulse it will oscillate as it orbits at a radius of $2a$. To determine the frequency at which the mass oscillates we write down Newton's law. There are two forces acting on the mass: the "centripetal" force and the spring force. Let us assume the mass has been displaced a distance Δ from $2a$. The masses acceleration can be written in terms of its frequency as $a' = \omega'^2(2a)$

$$\begin{aligned} \sum F =& ma \\ m\omega^2(2a+\Delta-a) + 2k(2a+\Delta-a) =& m\omega'^2(2a+\Delta-a) \\ \omega^2 + 2\frac{k}{m} =& \omega'^2 \\ \frac{\omega_0}{2} + 2\omega_0 =& \omega'^2 \\ \frac{\omega_0^2}{2} + \frac{2\omega_0^2}{2} =& \omega'^2 \\ \frac{5}{2}\omega_0^2 =& \omega'^2 \\ \sqrt{\frac{5}{2}}\omega_0 =& \omega' \end{aligned} \tag{34}$$

11 Extra Credit

11.1 Problem 3.16 Solution is Omitted

11.2 Problem 3.24

The motion of a particle in a plane may be described in terms of elliptic co-ordinates λ, θ defined by

$$x = Ccosh\lambda cos\theta, \qquad y = Csinh\lambda sin\theta, \qquad (\lambda \geq 0, 0 \leq \theta \leq 2\pi)$$

where c is a positive constant. Show that the kinetic energy function may be written

$$T = \frac{1}{2}mC^2\left(cosh^2\lambda - cos^2\theta\right)\left(\dot{\lambda}^2 + \dot{\theta}^2\right)$$

Hence write down the equations of motion.

11.2.1 Part A

The kinetic energy in Cartesian coordinates is $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. We will first need to calculate the time derivatives of x and y. Doing so results in the following two equations

$$\dot{x} = c\dot{\lambda}sinh\lambda cos\theta - c\dot{\theta}cosh\lambda sin\theta \qquad \dot{y} = c\dot{\lambda}cosh\lambda sin\theta + c\dot{\theta}sinh\lambda cos\theta$$

Squaring the x derivative term we obtain

$$\dot{x}^2 = c^2\dot{\lambda}^2sinh^2\lambda cos^2\theta - 2c^2\dot{\lambda}\dot{\theta}sinh\lambda cos\theta cosh\lambda sin\theta + c^2\dot{\theta}^2cosh^2\lambda sin^2\theta$$

And squaring the y derivative term we

$$\dot{y}^2 = c^2\dot{\theta}^2sinh^2\lambda cos^2\theta + 2c^2\dot{\lambda}\dot{\theta}sinh\lambda cos\theta cosh\lambda sin\theta + c^2\dot{\lambda}^2cosh^2\lambda sin^2\theta$$

Now the kinetic energy is just the sum of these two quantities and multiplied by $\frac{m}{2}$. Note that the middle terms cancel each other. We factor out a c^2 also.

$$\begin{aligned}
T =& \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)\\
T =& \frac{1}{2}mc^2\left(\dot{\lambda}^2sinh^2\lambda cos^2\theta + \dot{\theta}^2cosh^2\lambda sin^2\theta + \dot{\theta}^2sinh^2\lambda cos^2\theta + \dot{\lambda}^2cosh^2\lambda sin^2\theta\right)\\
T =& \frac{1}{2}mc^2\left[\left(\dot{\lambda}^2 + \dot{\theta}^2\right)\left(sinh^2\lambda cos^2\theta + cosh^2\lambda sin^2\theta + sinh^2\lambda cos^2\theta + cosh^2\lambda sin^2\theta\right)\right]\\
T =& \frac{1}{2}mc^2\left[2\left(\dot{\lambda}^2 + \dot{\theta}^2\right)\left(sinh^2\lambda cos^2\theta + cosh^2\lambda sin^2\theta\right)\right]\\
T =& \frac{1}{2}mc^2\left[\left(\dot{\lambda}^2 + \dot{\theta}^2\right)\left(cos^2\theta + cosh^2\lambda\right)\right] \qquad (35)
\end{aligned}$$

Where we have used the following identities in the last step of the derivation.

$$cosh^2(x) = \frac{1 + cosh(2x)}{2} \qquad sinh^2(x) = \frac{cosh(2x) - 1}{2} \qquad cosh(2x) = sinh^2(x) - cosh^2(x) \qquad sinh(2x) = 2sinh(x)cosh(x)$$

I evaluated the result with Matlab and concluded it was correct. The coordinate system is pictured below

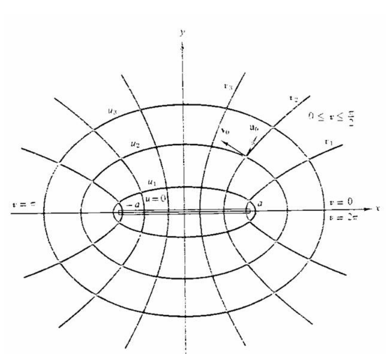


Figure 10: Elliptical coordinate representation.

11.2.2 Part B

If we wish to determine the equations of motion we shall extremize the Lagrangian. The Lagrangian is

$$L = T - V = \frac{1}{2}mc^2\left(\dot{\lambda}^2 + \dot{\theta}^2\right)\left(cos^2\theta + cosh^2\lambda\right) - V(\theta, \lambda)$$

Using the Euler-Lagrange Equations with the two independent parameters λ and θ. We will start the λ dependent equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}} = \frac{d}{dt}\left[2\dot{\lambda}\left(cos^2\theta + cosh^2\lambda\right)\right] = \frac{mc^2}{2}\left[2\ddot{\lambda}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sin(2\theta)\right]$$

$$\frac{\partial L}{\partial \lambda} = -mc^2 sinh(2\lambda)\left(\dot{\lambda}^2 - \dot{\theta}^2\right) - \frac{\partial V}{\partial \lambda}$$

Putting these two together, with the relation $F = -\frac{\partial V}{\partial \vec{r}}$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}} - \frac{\partial L}{\partial \lambda} &= 0 \\
\frac{mc^2}{2}\left[2\ddot{\lambda}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sin(2\theta)\right] - \left(-mc^2 sinh(2\lambda)\left(\dot{\lambda}^2 - \dot{\theta}^2\right) + \frac{\partial V}{\partial \lambda}\right) &= 0 \\
\frac{mc^2}{2}\left[2\ddot{\lambda}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sin(2\theta) + sinh(2\lambda)\left(\dot{\lambda}^2 - \dot{\theta}^2\right)\right] - \frac{\partial V}{\partial \lambda} &= 0 \\
\frac{mc^2}{2}\left[2\ddot{\lambda}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sin(2\theta) + sinh(2\lambda)\left(\dot{\lambda}^2 - \dot{\theta}^2\right)\right] &= \frac{\partial V}{\partial \lambda} = -F_\lambda \\
\frac{mc^2}{2}\left[2\ddot{\lambda}\left(cosh^2\lambda - cos^2\theta\right) + 2\dot{\lambda}\dot{\theta}sin(2\theta) - sinh(2\lambda)\left(\dot{\lambda}^2 - \dot{\theta}^2\right)\right] &= F_\lambda
\end{aligned} \tag{36}$$

Now we will determine the equations of motion from Euler-Lagrange equation for the angular dependence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left[2\dot{\theta}\left(cos^2\theta + cosh^2\lambda\right)\right] = \frac{mc^2}{2}\left[2\ddot{\theta}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sinh(2\lambda)\right]$$

$$\frac{\partial L}{\partial \theta} = mc^2 sin(2\theta)\left(\dot{\lambda}^2 - \dot{\theta}^2\right) - \frac{\partial V}{\partial \theta}$$

Putting these two together, with the relation $F = -\frac{\partial V}{\partial \vec{r}}$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\
\frac{mc^2}{2}\left[2\ddot{\theta}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sinh(2\lambda)\right] - \left(mc^2 sin(2\theta)\left(\dot{\lambda}^2 - \dot{\theta}^2\right) + \frac{\partial V}{\partial \theta}\right) &= 0 \\
\frac{mc^2}{2}\left[2\ddot{\theta}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sinh(2\lambda) + sin(2\theta)\left(\dot{\lambda}^2 - \dot{\theta}^2\right)\right] - \frac{\partial V}{\partial \theta} &= 0 \\
\frac{mc^2}{2}\left[2\ddot{\theta}\left(-cosh^2\lambda + cos^2\theta\right) - 2\dot{\lambda}\dot{\theta}sinh(2\lambda) + sin(2\theta)\left(\dot{\lambda}^2 - \dot{\theta}^2\right)\right] &= \frac{\partial V}{\partial \theta} = -F_\theta \\
\frac{mc^2}{2}\left[2\ddot{\theta}\left(cosh^2\lambda - cos^2\theta\right) + 2\dot{\lambda}\dot{\theta}sinh(2\lambda) - sin(2\theta)\left(\dot{\lambda}^2 - \dot{\theta}^2\right)\right] &= F_\theta
\end{aligned} \tag{37}$$

11.3 Problem 4.7

Calculate the period of a satellite in an orbit just above the Earths atmosphere (whose thickness may be neglected). Find also the periods for close orbits around the Moon and Jupiter.

11.3.1 Part A

We can use Kepler's law to calculate the orbital period of the satellite. The Earths radius is 6380km.

$$\tau = 2\pi\sqrt{\frac{a^3}{GM_E}} = 2\pi\sqrt{\frac{(6380km)^3}{G \cdot 5.98 \times 10^{24}kg}} = 5070s = 85min \tag{38}$$

11.3.2 Part B

We can use Kepler's law to calculate the orbital period of the satellite. The Moons radius is 0.273 times the earths and is about 0.0123 times the Earths mass.

$$\tau = 2\pi\sqrt{\frac{a^3}{GM_E}} = 2\pi\sqrt{\frac{(0.273 \cdot 6380km)^3}{0.0123 \cdot G \cdot 5.98 \times 10^{24}kg}} = 6479.6s = 108min \tag{39}$$

11.3.3 Part C

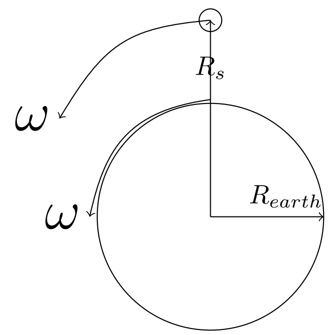
We can use Kepler's law to calculate the orbital period of the satellite. Jupiter's radius is 11.02 times bigger than earths, and is 318 times more massive.

$$\tau = 2\pi\sqrt{\frac{a^3}{GM_E}} = 2\pi\sqrt{\frac{(11.2\cdot 6380km)^3}{318G\cdot 5.98\times 10^{24}kg}} = 10,656s = 177.6min \tag{40}$$

12 Extra Problems

12.1 Problem 4.1

The orbits of synchronous communications satellites have been chosen so that viewed from the Earth they appear to be stationary. Find the radius of the orbits.



12.1.1 Solution

We know that earth and the satellite exert equal and opposite forces of each other, $F_{12} = -F_{21}$. This is the only force that could cause centripetal acceleration. For the earth and the satellite to have the same angular velocity they must have the same period of rotation. We know the earths period of rotation, T, as 24 hours, or 86,400 seconds. This means that the angular velocity of both should be $\omega = \frac{2\pi}{T}$. We also know that the force on the satellite is given by $F_s = G\frac{m_{earth}m_s}{r_s^2}$ and the acceleration is given by $a = \omega^2 r_s$. Putting this all together we obtain

$$\begin{aligned}
m_s\omega^2 r_s =& G\frac{m_{earth}m_s}{rs^2}\\
r_s^3 =& \frac{Gm_{earth}}{\omega^2}\\
r_s =& \left(\frac{Gm_{earth}}{\omega^2}\right)^{\frac{1}{3}}\\
r_s =& \left(\frac{T^2 Gm_{earth}}{4\pi^2}\right)^{\frac{1}{3}}\\
r_s =& 42,250km
\end{aligned} \tag{41}$$

12.2 Problem 4.2

Find the radii of synchronous orbits about Jupiter and about the Sun. Their mean rotation periods are 10 hours and 27 days, respectively. The mass of Jupiter is 318 times that of the Earth. The semi-major axis of the Earths orbit, or astronomical unit is $1.50\times 10^8 km$. Refer to problem 3.1 for a picture of the situation.

12.2.1 Part A

We will just make use of problem 3.1's solution. All we need is difference in mass of the object being orbited and the difference in rotational period. Jupiter's rotational period is $\frac{5}{12}$ times earths, and its mass is 318 times larger. Hence,

$$r_s = \left(\frac{(\frac{10}{24}T)^2 G318m_{earth}}{4\pi^2}\right)^{\frac{1}{3}} = 160,800km = 0.001AU \tag{42}$$

12.2.2 Part B

Jupiter's rotational period is 27 times earths, and its mass is 333,000 times larger. Hence,

$$r_s = \left(\frac{(27T)^2 G333,000m_{earth}}{4\pi^2}\right)^{\frac{1}{3}} = 26,356,300km = 0.176AU \tag{43}$$