Physics 105 Homework 2

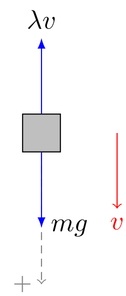
Eric Reichwein
Department of Physics
University of California, Santa Cruz

October 11, 2012

1 Homework 2 Problems

1.1 Problem 2.13

A particle falling under gravity is subject to a retarding force proportional to its velocity. Find its position as a function of time, if it starts from rest, and show that it will eventually reach a terminal velocity.



1.1.1 Part A

We first setup a differential equation using Newton's second and our free body diagram. We can see that if we write it in terms of velocity we have a first order separable differential equation.

$$
\begin{aligned}
\sum F =& ma \\
mg - \lambda v =& m\ddot{z} \\
m\dot{v} + \lambda v =& -mg \\
\frac{dv}{dt} =& -\gamma v - g \\
\int_0^v \frac{dv}{\gamma v + g} =& -\int_0^t dt \\
\frac{1}{g}\int_0^v \frac{dv}{\frac{\gamma}{g}v + 1} =& -\int_0^t dt \\
\frac{1}{\gamma}\frac{g}{\gamma} ln\left[\frac{\gamma}{g}v + 1\right] =& -t \\
ln\left[\frac{\gamma}{g}v + 1\right] =& -\gamma t \\
\frac{\gamma}{g}\dot{z} + 1 =& e^{-\gamma t} \\
\gamma\dot{z} + g =& ge^{-\gamma t} \\
\dot{z} =& g\frac{e^{-\gamma t} - 1}{\gamma}
\end{aligned}
$$

Now we have another first order separable differential equation that we can integrate to get position as a function of time.

$$\begin{aligned}
\frac{dz}{dt} =& ge^{-\gamma t} - g \\
\int_0^z dz =& \int_0^t dt \frac{ge^{-\gamma t} - g}{\gamma} \\
z =& \frac{1}{\gamma}\int_0^t dt ge^{-\gamma t} - \frac{g}{\gamma}\int_0^t dt \\
z =& \frac{1}{\gamma}\left[\frac{ge^{-\gamma t}}{-\gamma}\right]_0^t - \frac{gt}{\gamma} \\
z =& \frac{1}{\gamma}\left[-\frac{ge^{-\gamma t}}{\gamma} - \left(-\frac{ge^{-\gamma 0}}{\gamma}\right)\right] - \frac{gt}{\gamma} \\
z =& \frac{1}{\gamma}\left[-\frac{ge^{-\gamma t}}{\gamma} + \frac{g}{\gamma}\right] - \frac{gt}{\gamma} \\
z =& \frac{g}{\gamma^2}\left(1 - ge^{-\gamma t}\right) - \frac{gt}{\gamma} \qquad (1)
\end{aligned}$$

1.1.2 Part B

We can see that the particle has a limiting speed which is found by taking time to infinity for the velocity function

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} g\frac{e^{-\gamma t} - 1}{\gamma} = \frac{e^{-\gamma\infty} - 1}{\gamma} = \frac{g}{\gamma} \qquad (2)$$

1.2 Problem 2.14

The terminal speed of the particle in Problem 13 is $50\frac{m}{s}$. Find the time it takes to reach a speed of $40\frac{m}{s}$, and the distance it has fallen in that time.

1.2.1 Part A

Since the terminal velocity of an object is $50\frac{m}{s}$ we can determine the coefficient of the retarding force, γ, because $v_{terminal} = \frac{g}{\gamma} \longrightarrow \gamma = \frac{1}{5}$. Now we just use our velocity equation found in the previous problem to obtain t.

$$\begin{aligned}
-40\frac{m}{s} =& \frac{ge^{-\gamma t} - g}{\gamma} \\
-\frac{40\frac{m}{s}\gamma}{g} + 1 =& e^{-\gamma t} \\
ln\left[-\frac{40\frac{m}{s}\gamma}{g} + 1\right] =& -\gamma t \\
t =& -\frac{ln\left[-40\frac{m}{s}\gamma + 1\right]}{\gamma} \\
t =& -5ln\left[-40\frac{m}{s}\frac{1}{10\cdot 5} + 1\right] \\
t =& -5ln\left[0.2\right] \\
t =& 8.05s \qquad (3)
\end{aligned}$$

1.2.2 Part B

To determine how far the object fell we just plug out time into position function

$$\begin{aligned}
z =& \frac{g}{\gamma^2}\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma} \\
z =& \frac{10}{\frac{1}{5}^2}\left(1 - e^{-\frac{8.05}{5}}\right) - 10\cdot 8.05\cdot 5 \\
z =& 250\left(1 - 0.2\right) - 402.5 \\
z =& 200 - 402.5 \\
z =& -202.5m \qquad (4)
\end{aligned}$$

1.3 Problem 2.20

A particle of mass m moves in the region $x > 0$ under the force $F = -m\omega^2(x - \frac{a^4}{x^3})$, where ω and a are constants. Sketch the potential energy function. Find the position of equilibrium, and the period of small oscillations about it. The particle starts from this point with velocity v. Find the limiting values of x in the subsequent motion. Show that the period of oscillation is independent of v.

1.3.1 Part A

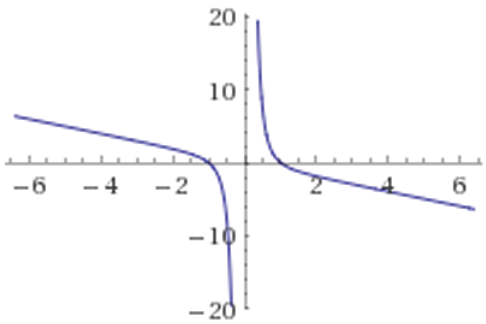


Figure 1: The force function $F(x) = -m\omega^2\left(1 - \frac{a^4}{x^3}\right)$

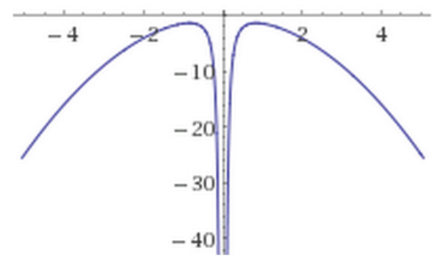


Figure 2: The potential function $V(x) = -m\omega^2\left(\frac{x^2}{2} + \frac{a^4}{2x^2}\right)$

1.3.2 Part B

To find the equilibrium position we must set the force equal to zero

$$\begin{aligned} F = 0 =& -m\omega^2\left(x - \frac{a^4}{x^3}\right) \\ x =& \frac{a^4}{x^3} \\ x^4 =& a^4 \\ x =& \pm a \end{aligned} \tag{5}$$

To find the period of oscillation we first need to compute $V'' = F' = m\omega^2\left(1 + \frac{3a^4}{x^4}\right)$. The equation for period is $T = 2\pi\sqrt{\frac{m}{V''(x)}}$. Also, V'' evaluated at $x = \pm a$, is $V''(x = \pm a) = 4m\omega^2$. Hence, the period of oscillation is

$$T = 2\pi\sqrt{\frac{m}{4m\omega^2}} = \frac{2\pi}{2\omega} = \frac{\pi}{\omega} \tag{6}$$

1.3.3 Part C

If a particle starts at this point and is given a velocity v the limiting x positions will be found by setting its kinetic energy to the potential energy.

$$\begin{aligned} T_i + V_i =& V_f \\ \frac{mv^2}{2} + -m\omega^2\left(\frac{a^2}{2} + \frac{a^4}{2a^2}\right) =& -m\omega^2\left(\frac{x^2}{2} + \frac{a^4}{2x^2}\right) \\ \frac{mv^2}{2} - m\omega^2 a^2 =& -m\omega^2\left(\frac{x^2}{2} + \frac{a^4}{2x^4}\right) \\ \frac{v^2}{2\omega^2} - a^2 =& -\frac{x^2}{2} - \frac{a^4}{2x^2} \\ \left(\frac{v^2}{\omega^2} - 2a^2\right)x^2 =& -x^4 - a^4 \\ x^4 + \left(\frac{v^2}{\omega^2} - 2a^2\right)x^2 + a^4 =& 0 \end{aligned}$$

Now we make the subsitution $y = x^2$. I solved the resulting bi-quadratic equation on MatLab but the steps to solving are as follows

$$
\begin{aligned}
y^2 + \left(\frac{v^2}{\omega^2} - 2a^2\right) y + a^4 &= 0 \\
y &= \frac{-\left(\frac{v^2}{\omega^2} - 2a^2\right) \pm \sqrt{\left(\frac{v^2}{\omega^2} - 2a^2\right)^2 - 4(1)a^4}}{2(1)} \\
y &= \frac{-\left(\frac{v^2}{\omega^2} - 2a^2\right) \pm \sqrt{\left(\frac{v^4}{\omega^4} - 4\frac{v^2}{\omega^2}a^2 + 4a^4\right) - 4a^4}}{2} \\
y &= \frac{-\left(\frac{v^2}{\omega^2} - 2a^2\right) \pm \sqrt{\frac{v^4}{\omega^4} - 4\frac{v^2}{\omega^2}a^2}}{2} \\
y &= \frac{-\left(\frac{v^2}{\omega^2} - 2a^2\right) \pm 2\frac{v}{\omega}\sqrt{\frac{v^2}{4\omega^2} - a^2}}{2} \\
y &= -\left(\frac{v^2}{2\omega^2} - a^2\right) \pm \frac{v}{\omega}\sqrt{\frac{v^2}{4\omega^2} - a^2}
\end{aligned}
$$

Now we will just take the square root of this equation to obtain the roots of x of the bi-quadratic equation.

$$
\begin{aligned}
y = x^2 &= -\frac{v^2}{2\omega^2} \pm \frac{v}{\omega}\sqrt{\frac{v^2}{4\omega^2} - a^2} + a^2 \\
x &= \pm\sqrt{-\frac{v^2}{2\omega^2} \pm \frac{v}{\omega}\sqrt{\frac{v^2}{4\omega^2} - a^2} + a^2} \\
x &= \sqrt{\frac{v^2}{4\omega^2} + a^2} \pm \frac{v^2}{2\omega} \qquad (7)
\end{aligned}
$$

Where the last step was computed using Wolfram Alpha.

1.3.4 Part D

Proof. To see that the period is independent of velocity we can just use the equation of period

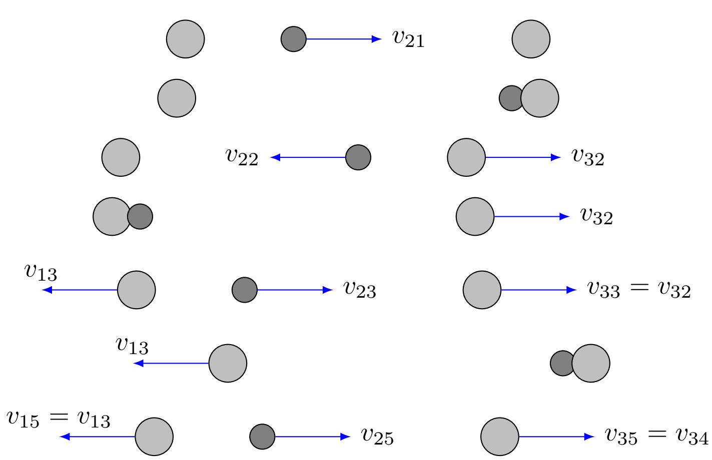
$$\tau = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{V''(x)}} = 2\pi\sqrt{\frac{m}{F'(t)}} = 2\pi\sqrt{\frac{m}{ma'(t)}} = 2\pi\sqrt{\frac{1}{a'(t)}}$$

Hence the period is dependent on the rate of change of the acceleration not on velocity.

□

1.4 Problem 2.27

Three perfectly elastic bodies of masses 5 kg, $1kg$, $5kg$ are arranged in that order on a straight line, and are free to move along it. Initially, the middle one is moving with velocity $27\frac{m}{s}$, and the others are at rest. Find how many collisions take place in the subsequent motion, and verify that the final value of the kinetic energy is equal to the initial value.



1.4.1 Initial energy

There is only kinetic energy in the initial situation given by

$$E = T_i = \frac{m_2 v_{21}^2}{2} = 364.5J$$

1.4.2 First Collision

To determine the resulting velocities v_{22} and v_{32}. Since it is an elastic collision we can say energy is also conserved. We first write down the equations for conservation of momentum and conservation of energy.

$$\begin{aligned} P_{1i} &= P_{1f} \\ m_2 v_{21} &= -m_2 v_{22} + m_3 v_{32} \\ m_2 v_{21} + m_2 v_{22} &= m_3 v_{32} \\ m_2 \left(v_{21} + v_{22}\right) &= m_3 v_{32} \end{aligned} \tag{8}$$

$$\begin{aligned} E_{1i} &= E_{1f} \\ \frac{1}{2} m_2 v_{21}^2 &= \frac{1}{2} m_2 v_{22}^2 + \frac{1}{2} m_3 v_{32}^2 \\ \frac{1}{2} m_2 v_{21}^2 - \frac{1}{2} m_2 v_{22}^2 &= \frac{1}{2} m_3 v_{32}^2 \\ m_2 \left(v_{21}^2 - v_{22}^2\right) &= \frac{1}{2} m_3 v_{32}^2 \\ m_2 \left(v_{21} - v_{22}\right)\left(v_{21} + v_{22}\right) &= m_3 v_{32}^2 \end{aligned} \tag{9}$$

1.4.3 Second Collision

Now we divide the energy equation by the momentum equation to obtain $v_{21} - v_{22} = v_{32}$. Plugging this back into the momentum equation we get

$$\begin{aligned} m_2 \left(v_{21} + v_{22}\right) &= m_3 v_{32} \\ m_2 v_{21} + m_2 v_{22} &= m_3 v_{21} - m_3 v_{22} \\ -m_3 v_{22} - m_2 v_{22} &= m_2 v_{21} - m_3 v_{21} \\ v_{22} &= \frac{\left(m_3 - m_2\right) v_{21}}{\left(m_3 + m_2\right)} \\ v_{22} = -\frac{2}{3} v_{21} &= -\frac{2}{3} 27 \frac{m}{s} = -18 \frac{m}{s} \end{aligned}$$

We can then find v_{32} by $v_{32} = v_{21} - v_{22} = 9\frac{m}{s}$. The second ball then begins traveling towards the first ball. We have the same situation with a ball traveling with initial velocity then striking a stationary ball. We can use the same equations as derived previously to find the resulting velocities. The equation for velocities of this new situation is $v_{13} = v_{22} + v_{23}$. Plugging back into the momentum equation we obtain

$$v_{23} = \frac{\left(m_1 - m_2\right) v_{22}}{\left(m_1 + m_2\right)} = \frac{2}{3} v_{22} = 12 \frac{m}{s}$$

After the collision ball 1 will have velocity $v_{13} = v_{22} + v_{23} = 12\frac{m}{s} - 18\frac{m}{s} = -6\frac{m}{s}$. Ball 2 will hit ball 3 again because ball 2 has a larger velocity than ball 3 ($v_{23} > v_{32}$).

1.4.4 Third Collision

Since now we have a collision between two moving particles we must derive an equation for the resulting velocities using the same technique.

$$\begin{aligned} P_{2i} &= P_{2f} \\ m_2 v_{23} + m_3 v_{33} &= m_2 v_{24} + m_3 v_{34} \\ m_2 v_{23} - m_2 v_{24} &= -m_3 v_{33} + m_3 v_{34} \\ m_2 \left(v_{23} - v_{24}\right) &= m_3 \left(v_{34} - v_{33}\right) \end{aligned} \tag{10}$$

$$\begin{aligned} E_{2i} &= E_{2f} \\ \frac{1}{2} m_2 v_{23}^2 + \frac{1}{2} m_3 v_{33}^2 &= \frac{1}{2} m_2 v_{24}^2 + \frac{1}{2} m_3 v_{34}^2 \\ \frac{1}{2} m_2 v_{23}^2 - \frac{1}{2} m_2 v_{24}^2 &= \frac{1}{2} m_3 v_{34}^2 - \frac{1}{2} m_3 v_{33}^2 \\ m_2 \left(v_{23}^2 - v_{24}^2\right) &= m_3 \left(v_{34}^2 - v_{33}^2\right) \\ m_2 \left(v_{23} - v_{24}\right)\left(v_{23} + v_{24}\right) &= m_3 \left(v_{34} - v_{33}\right)\left(v_{34} + v_{33}\right) \end{aligned} \tag{11}$$

Dividing these equations and then plugging back into momentum equation we obtain

$$\frac{m_2(v_{23}-v_{24})(v_{23}+v_{24})}{m_2(v_{23}-v_{24})} = \frac{m_3(v_{34}-v_{33})(v_{34}+v_{33})}{m_3(v_{34}-v_{33})}$$
$$v_{23}+v_{24} = v_{34}+v_{33}$$
$$v_{34} = v_{23}+v_{24}-v_{33}$$

$$m_2v_{23}+m_2v_{24} = m_3v_{33}-m_3v_{34}$$
$$m_2v_{23}+m_2v_{24} = m_3v_{33}-m_3(v_{23}+v_{24}-v_{33})$$
$$m_2v_{23}+m_2v_{24} = m_3v_{33}-m_3v_{23}-m_3v_{24}+m_3v_{33}$$
$$m_3v_{24}+m_2v_{24} = m_3v_{33}-m_2v_{23}-m_3v_{24}+m_3v_{33}$$
$$(m_2+m_3)v_{24} = m_3v_{33}-m_2v_{23}-m_3v_{23}+m_3v_{33}$$
$$v_{24} = \frac{m_3v_{33}-m_2v_{23}-m_3v_{23}+m_3v_{33}}{(m_3+m_2)}$$
$$v_{24} = \frac{2m_3v_{33}-(m_3+m_2)v_{23}}{(m_3+m_2)}$$

Plugging the numbers in we obtain $v_{24} = 7\frac{m}{s}$ and $v_{34} = v_{23}+v_{24}-v_{33} = 9\frac{m}{s}+7\frac{m}{s}-6\frac{m}{s} = 10\frac{m}{s}$

1.4.5 Final Energy and Number of Collisions

The final energy is given by the sum of the individual kinetic energies.

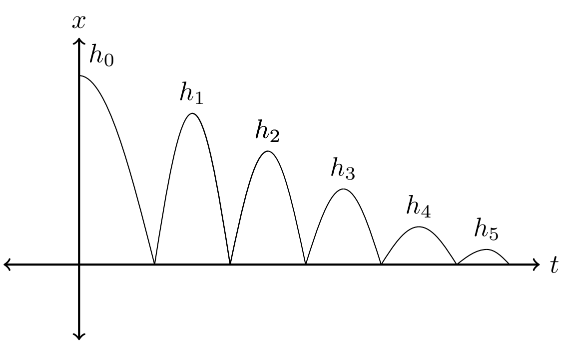
$$T_f = \frac{m_1v_{14}^2}{2}+\frac{m_2v_{24}^2}{2}+\frac{m_3v_{34}^2}{2} = \frac{5kg\cdot 6\frac{m}{s}^2}{2}+\frac{1kg\cdot 7\frac{m}{s}^2}{2}+\frac{5kg\cdot 10\frac{m}{s}^2}{2} = 364.5J$$

We also note there was only 3 collisions.

1.5 Problem 2.28

A ball is dropped from height h and bounces. The coefficient of restitution at each bounce is e. Find the velocity immediately after the first bounce, and immediately after the nth bounce. Show that the ball finally comes to rest after a time

$$\frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$$



1.5.1 Part A

The ball is dropped from height h. To find the velocity immediately after the first bounce we must first calculate the the velocity of the ball just before it hits the ground. Then we use the definition of the coefficient of restitution, $e = -\frac{v_1}{v_2}$. We use energy conservation since energy is conserved until the ball hits the ground, where the collision is inelastic.

$$E_i = E_f$$
$$mgh_1+\frac{1}{2}mv_1^2 = mgh+\frac{1}{2}mv_2^2$$
$$mgh = \frac{1}{2}mv_2^2$$
$$v_2^2 = gh$$
$$v_2 = \sqrt{2gh}$$

This is the velocity of the ball just before it has the inelastic collision with the ground, now to find the velocity just after it hits the ground we just multiply it by the coefficient of restitution

$$v_2 = -ev_1 = -e\sqrt{2gh} \tag{12}$$

1.5.2 Part B

To find the velocity after the $n-th$ bounce we need to find a relation between each bounce. The velocity after the first bounce is $v_2 = -e\sqrt{2gh}$. The ball then loses no energy until the second bounce where its velocity just before it hits the ground is v_2 (since no energy is lost until it hits the ground). We then know that the velocity after the third bounce is $v_3 = ev_2 = e^2\sqrt{2gh}$. We see that this continues for each bounce hence

$$v_n = -e^n\sqrt{2gh} \tag{13}$$

Where I have defined downward as positive.

1.5.3 Part C

To find the time it takes the ball to come to a stop we first need to compute how much distance it traveled. It traveled a distance h then bounces and travels $h_1 = e^2 h$ to the apex of its trajectory. The height of the second bounce is $h_2 = e^2 h_1 = e^4 h$. The ball travels both upward a distance h_n then also travels back down a distance h_n. We see this continues forever. This means we have a infinite series. The total distance traveled is given by

$$h_t = -h+2h+2e^2h+2e^4h+\cdots+2e^{2n}h = -h+h\sum_{n=1}^{\infty}2^n e^{2n} = -h+\lim_{n\to\infty}2h\frac{1-e^{2n}}{1-e^2} = \frac{2h}{1-e^2}-h\frac{1-e^2}{1-e^2} = \frac{2h-h\left(1-e^2\right)}{1-e^2} = \frac{h(1+e^2)}{1-e^2}$$

Now using the position function $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$, with $x_0 = 0$, $x(t) = h_n$ and $v_0 = 0$. we will obtain

$$\begin{aligned}
h_n =& 0 + 0\cdot t + \frac{1}{2}gt^2 \\
t^2 =& 2\frac{h_n}{g} \\
t^2 =& 2\frac{\frac{h(1+e^2)}{1-e^2}}{g} \\
t =& \sqrt{2\frac{\frac{h(1+e^2)}{1-e^2}}{g}} \\
t =& \frac{1+e}{1-e}\sqrt{\frac{2h}{g}}
\end{aligned} \tag{14}$$

1.6 Problem 2.32

Find the Greens function of an oscillator in the case $\gamma > \omega_0$. Use it to solve the problem of an oscillator that is initially in equilibrium, and is subjected from $t = 0$ to a force increasing linearly with time, $F = ct$.

1.6.1 Part A

The solution to the over damped oscillator is

$$x(t) = Ae^{-\gamma_+ t} + Be^{-\gamma_- t}$$

Where $\gamma_+ = \gamma + \sqrt{\gamma^2 - \omega_0^2}$ and $\gamma_- = \gamma - \sqrt{\gamma^2 - \omega_0^2}$. To find the constants A and B we use the fact that $t \leq 0$ the oscillator is in equilibrium, and γ_+ is the positive root and γ_- is the negative root. When plugging back into the equation we find that

$$\begin{aligned}
A(\gamma_+^2 + \gamma\gamma_+ + \omega_0^2) - B(\gamma_-^2 + \gamma\gamma_- + \omega_0^2) =& 0 \\
A - B =& 0 \\
A =& B
\end{aligned}$$

To find the value of A or B we plug initial values in to the velocity function, which is obtained by differentiating the position function to obtain that

$$A = \frac{I_r}{m(\gamma_+ - \gamma_-)} = B$$

Hence, the solution to the homogeneous equation is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ \dfrac{I}{m}\dfrac{e^{-\gamma_+ t} + e^{-\gamma_- t}}{\gamma_+ - \gamma_-} & t \geq 0 \end{cases}$$

Now consider an impulse I_r at $t = t'$ we can then define Greens function as

$$G(t-t') = \begin{cases} 0 & t < t' \\ \dfrac{I_r}{m}\dfrac{e^{-\gamma_+(t-t')} + e^{-\gamma_-(t-t')}}{\gamma_+ - \gamma_-} & t \geq t' \end{cases}$$

1.6.2 Part B

Now by using Greens theorem we can find the response to the force.

$$
\begin{aligned}
x(t) =& \int_0^t G(t-t')F(t)dt' \\
=& c\int_0^t \frac{1}{m}\frac{e^{-\gamma_+(t-t')}+e^{-\gamma_-(t-t')}}{\gamma_+-\gamma_-}tdt' \\
=& \frac{c}{m(\gamma_+-\gamma_-)}\int_0^t \left[e^{-\gamma_+(t-t')}+e^{-\gamma_-(t-t')}\right]t'dt' \\
=& \frac{I}{m(\gamma_+-\gamma_-)}\left[\int_0^t te^{-\gamma_+(t-t')}dt' + \int_0^t te^{-\gamma_-(t-t')}dt'\right] \\
x(t) =& \frac{c}{m(\gamma_+-\gamma_-)}\left[\frac{1-e^{-\gamma_+(t)}(\gamma_+ t+1)}{\gamma_+^2}+\frac{1-e^{-\gamma_-(t)}(\gamma_- t+1)}{\gamma_-^2}\right]
\end{aligned}
$$

When algebraically manipulated further we obtain the result

$$
x(t) = \frac{c}{m}\left[\frac{1}{(\gamma_+-\gamma_-)}\left(\frac{e^{-\gamma_+(t)}}{\gamma_+^2}+\frac{e^{-\gamma_+(t)}}{\gamma_+^2}\right)-\frac{2\gamma}{\omega^4}+\frac{t}{\omega^2}\right] \tag{15}
$$

Where we have previously defined $\omega_0^2 = \frac{k}{m}$ and $\gamma = \frac{2\lambda}{m}$. I used the integral formula

$$
\int te^{-bt}dt = \frac{e^{-bt}(bt+1)}{b^2}
$$

which I found in a integral table from a introductory calculus textbook.

1.7 Problem 3.1

Find which of the following forces are conservative, and for those that are find the corresponding potential energy function (a and b are constants, and a is a constant vector): (a) $F_x = ax + by^2$, $F_y = az + 2bxy$, $F_z = ay + bz^2$; (b) $F_x = ay$, $F_y = az$, $F_z = ax$; (c) $F_r = 2arsin\theta sin\phi$, $F_\theta = arcos\theta sin\phi$, $F_\phi = arcos\phi$; (d) $F = a \wedge r$; (e) $F = ra$; (f) $F = a(a \cdot r)$.

1.7.1 Defining What Conservative Means

To determine whether functions are conservative we must compute the curl of the function if it equals zero then the function is conservative.

$$
\vec{\nabla}\wedge\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right) - \hat{j}\left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)
$$

1.7.2 Part A

$$
\begin{aligned}
\frac{\partial F_z}{\partial x} &= 0 & \frac{\partial F_x}{\partial z} &= 0 & \frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z} &= 0-0=0 \\
\frac{\partial F_y}{\partial x} &= 2by & \frac{\partial F_x}{\partial y} &= 2by & \frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z} &= 2by-2by=0 \\
\frac{\partial F_y}{\partial z} &= a & \frac{\partial F_z}{\partial y} &= a & \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} &= a-a=0
\end{aligned}
$$

This this force is **conservative**. The associated potential function is given by $V(\vec{r}) = -\int^{\vec{r}}\vec{F}\cdot d\vec{r} = -\int^x F_x dx - \int^y F_y dy - \int^z F_z dz$ which applied to our force function gives

$$
V(x) = -\frac{1}{2}ax^2 - by^2x - azy - bxy^2 - ayz - \frac{1}{3}bz^3 = -\frac{1}{2}ax^2 - 2by^2x - 2ayz - \frac{1}{3}bz^3 \tag{16}
$$

1.7.3 Part B

$$
\begin{aligned}
\frac{\partial F_z}{\partial x} &= a & \frac{\partial F_x}{\partial z} &= 0 & \frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z} &= a-0=a \\
\frac{\partial F_y}{\partial x} &= 0 & \frac{\partial F_x}{\partial y} &= a & \frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z} &= 0-a=-a \\
\frac{\partial F_y}{\partial z} &= a & \frac{\partial F_z}{\partial y} &= 0 & \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} &= a-0=a
\end{aligned}
$$

Hence, $\vec{\nabla}\wedge\vec{F} \neq 0$ so this force is **non-conservative.**

1.7.4 Part C

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \frac{\hat{1}_r}{r^2 sin\phi} & \frac{\hat{1}_\phi}{r sin\phi} & \frac{\hat{1}_\theta}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_r & rF_\phi & rsin\phi F_\theta \end{vmatrix} = \frac{\hat{1}_r}{r^2 sin\phi}\left(\frac{\partial F_\theta}{\partial \phi} rsin\phi - \frac{\partial F_\phi}{\partial \theta} r\right) - \frac{\hat{1}_\phi}{rsin\phi}\left(\frac{\partial F_\theta}{\partial r} rsin\phi - \frac{\partial F_r}{\partial \theta}\right) + \frac{\hat{1}_\theta}{r}\left(\frac{\partial F_\phi}{\partial r} r - \frac{\partial F_r}{\partial \phi}\right)$$

I evaluated this using Matlab using the code:

```
curl([2*a*r*sin(theta)*sin(phi), a*r*cos(phi)*sin(theta), a*r*cos(phi)], [r, phi, theta], Spherical)
```

ans = 0 0 0

Hence, $\vec{\nabla} \wedge \vec{F} \neq 0$ so this force is **conservative.**

The corresponding potential function was also found using Matlab and was determined to be

$$V = -ar^2 sin\theta sin\phi \tag{17}$$

1.7.5 Part D

We will use the identity $\vec{\nabla} \wedge (\vec{a} \wedge \vec{r}) = [\vec{\nabla} \cdot \vec{a} + \vec{a} \cdot \vec{\nabla}]\vec{r} - [\vec{\nabla} \cdot \vec{r} + \vec{r} \cdot \vec{\nabla}]\vec{a}$ to compute the curl of a wedge product. Since in general $\vec{\nabla} \cdot \vec{G} = \vec{G} \cdot \vec{\nabla}$ we can see that

$$\begin{aligned} \vec{\nabla} \wedge (\vec{a} \wedge \vec{r}) &= [\vec{\nabla} \cdot \vec{a} + \vec{a} \cdot \vec{\nabla}]\vec{r} - [\vec{\nabla} \cdot \vec{r} + \vec{r} \cdot \vec{\nabla}]\vec{a} \\ &= [2\vec{\nabla} \cdot \vec{a}]\vec{r} - [2\vec{\nabla} \cdot \vec{r}]\vec{a} \\ &\neq 0 \end{aligned}$$

Hence, $\vec{\nabla} \wedge \vec{F} \neq 0$ so this force is **non-conservative.**

1.7.6 Part E

$$\frac{\partial F_z}{\partial x} = \frac{da_z}{dx} \qquad \frac{\partial F_x}{\partial z} = \frac{da_x}{dz} \qquad \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{da_z}{dx} - \frac{da_x}{dz} \neq 0$$

$$\frac{\partial F_y}{\partial x} = \frac{da_y}{dx} \qquad \frac{\partial F_x}{\partial y} = \frac{da_x}{dy} \qquad \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = \frac{da_y}{dx} - \frac{da_x}{dy} \neq 0$$

$$\frac{\partial F_y}{\partial z} = \frac{da_y}{dz} \qquad \frac{\partial F_z}{\partial y} = \frac{da_z}{dy} \qquad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{da_y}{dz} - \frac{da_z}{dy} \neq 0$$

Hence, $\vec{\nabla} \wedge \vec{F} \neq 0$ so this force is **non-conservative.**

1.7.7 Part F

If we try to take the curl of a scalar we will quickly see it is zero. A dot product is a scalar, hence

$$\vec{\nabla} \wedge [\vec{a}(\vec{a} \cdot \vec{r})] = 0$$

This this force is **conservative**. The associated potential function is given by $V(\vec{r}) = -\int^{\vec{r}} \vec{F} \cdot d\vec{r}'$ which applied to our force function gives

$$V(\vec{r}) = -\frac{1}{2}(\vec{a} \cdot \vec{r})^2 \tag{18}$$

1.8 Problem 3.2

Given that the force is as in Problem $1(a)$, evaluate the work done in taking a particle from the origin to the point $(1, 1, 0)$: (i) by moving first along the x-axis and then parallel to the y-axis, and (ii) by going in a straight line. Verify that the result in each case is equal to minus the change in the potential energy function.

1.8.1 Part i

The force is $F_x = ax + by^2$, $F_y = az + 2bxy$, $F_z = ay + bz^2$. We will parameterize the path from first to point $r_1 = 1, 0, 0)$ using $(t, 0, 0)$ then to $r_2 = (1, 1, 0)$ using $(1, t, 0)$. The first change in potential is

$$V(r) = -\int^{r_1} \vec{F} \cdot (dt, 0, 0) = -\int^{1} at + by^2 dt = \frac{at^2}{2} - by^2 t|_{r_0}^{r_1} = -\frac{a}{2}$$

Then we integrate from $r_1 = 1, 0, 0)$ to $r_2 = (1, 1, 0)$ using $(1, t, 0)$. We then obtain

$$V(r) = -\int_{r_1}^{r_2} \vec{F} \cdot (0, dt, 0) = -\int_{r_1}^{r_2} az + 2bxt \cdot dt = azt + bxt^2|_{r_1}^{r_2} = -b$$

Adding these two values together we obtain

$$\Delta V = -\frac{a}{2} - b = -\frac{a}{2} - b \tag{19}$$

1.8.2 Part *ii*

We now will take a straight line path parameterized by $(t, t, 0)$. We find that the line integral is then

$$\Delta V = -\int_{r_0}^{r_2} F \cdot (dt, dt, 0) = -\int_0^1 at + by^2 dt - \int_0^1 az + 2bxtdt = -\frac{a}{2} - 0 - 0 - b - 0 = -\frac{a}{2} - b$$

1.8.3 Part *iii*

To make sure these results agree with our potential function we will plug in the values directly to the potential function

$$\Delta V = V(1,1,0) - V(0,0,0) = \left[-\frac{a}{2} - 0 - b - 0\right] - [0 - 0 - 0 - 0] = -\frac{a}{2} - b$$

1.9 Problem 3.4

Compute the work done in taking a particle around the circle $x2 + y2 = a2$, $z = 0$ if the force is (a) $F_1 = y\hat{i}$, and (b) $F_2 = x\hat{i}$. What do you conclude about these forces?

1.9.1 Part A

We will parameterize our curve with $c(\theta) = a(cos\theta, sin\theta)$, which means $c'(\theta) = a(-sin\theta, cos\theta)d\theta$. Using the work energy theorem we conclude that

$$\begin{aligned} W = -\Delta V =& -a\int_c F_1 c'(\theta)d\theta \\ =& -a\int_0^{2\pi} sin\theta(-sin\theta)d\theta \\ =& a\int_0^{2\pi} sin^2\theta d\theta \\ =& \left[\frac{a\theta}{2} - \frac{asin(2\theta)}{2}\right]_0^{2\pi} \\ =& a\pi \end{aligned} \tag{20}$$

1.9.2 Part B

We will parameterize our curve with $c(\theta) = a(cos\theta, sin\theta)$, which means $c'(\theta) = a(-sin\theta, cos\theta)d\theta$. Using the work energy theorem we conclude that

$$\begin{aligned} W = -\Delta V =& -a\int_c F_2 c'(\theta)d\theta \\ =& -a\int_0^{2\pi} cos\theta(-sin\theta)d\theta \\ =& a\int_0^{2\pi} sin(2\theta)\theta d\theta \\ =& -a\left[\frac{cos(2\theta)}{2}\right]_0^{2\pi} \\ =& 0 \end{aligned} \tag{21}$$

1.9.3 Part C

The first force is non conservative and the second force is conservative.

1.10 Problem 3.5

Evaluate the force corresponding to the potential energy function $V(r) = \frac{cz}{r^3}$, where c is a constant. Write your answer in vector notation, and also in spherical polars, and verify that it satisfies $\vec{\nabla} \wedge \vec{F} = 0$.

1.10.1 Part A

We first note that the negative gradient of the potential is the force function. To determine the force function in vector notation we use $r = \sqrt{x^2 + y^2 + z^2}$. From here we see that we just differentiate with respect to x, y, and z.

$$\begin{aligned}
\vec{F} =& -\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \\
=& -\left(-\frac{3cxz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, -\frac{3cyz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, -\frac{c}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3cz^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) \\
=& c\left(\frac{3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \frac{3yz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) \\
=& c\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{r^2-3z^2}{r^5}\right) \\
=& \frac{c}{r^5}\left(3xz, 3yz, r^2 - 3z^2\right) \\
=& \frac{c}{r^5}\left(3xz, 3yz, x^2+y^2+z^2 - 3z^2\right) \\
=& \frac{c}{r^5}\left(3zx\hat{i}, 3zy\hat{j}, x^2+y^2+z^2 - 2z^2\hat{k}\right) \\
=& \frac{c}{r^5}\left(3\left(\vec{r}z\right) - z^2\hat{k}\right) \\
=& \frac{c}{r^5}\left(3\left(\vec{r}(\vec{r}\cdot\hat{k})\right) - r^2\cdot\hat{k}\right)
\end{aligned} \tag{22}$$

1.10.2 Part B

To write this in spherical polar we must make the change $x = rsin\theta cos\phi$, $y = rsin\theta sin\phi$, $z = rcos\theta$.

$$\begin{aligned}
\vec{F} =& -\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \\
=& -\left(-\frac{3cxz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, -\frac{3cyz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, -\frac{c}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{3cz^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) \\
=& c\left(\frac{3xz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \frac{3yz}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\right) \\
=& c\left(\frac{2cos\theta}{r^3}, \frac{2sin\theta}{r^3}, 0\right)
\end{aligned} \tag{23}$$

1.10.3 Part C

Given that the curl is defined by

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{j}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

I evaluated the curl in MatLab and concluded that

$$\vec{\nabla} \wedge \vec{F} = 0 \longrightarrow \text{F is conservative}$$

2 Extra Credit Problems

2.1 Problem 2.30

A particle moving under a conservative force oscillates between x_1 and x_2. Show that the period of oscillation is

$$\tau = 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2(V(x_2) - V(x_1))}}dx$$

In particular, if $V = \frac{1}{2}m\omega_0^2(x^2bx^4)$, show that the period for oscillations of amplitude a is

$$\tau = \frac{2}{\omega}\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}\sqrt{1-b(a^2+x^2)}}$$

Using the binomial theorem to expand in powers of b, and the substitution $x = asin\theta$, show that for small amplitude the period is approximately

$$\tau \approx \frac{2\pi}{\omega_0}(1 + \tfrac{3}{4}ba^2)$$

2.1.1 Part A

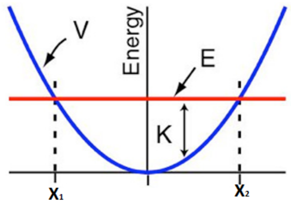


Figure 3: The potential function $V(x) = -m\omega^2\left(\frac{x^2}{2} + \frac{a^4}{2x^2}\right)$

Using energy conservation, and the picture above we see that the velocity of the particle at any point which lies in between $x_1 < x < x_2$ is found by

$$\begin{aligned} T + V =& E = V(x_2) = V(x_1) \\ \frac{mv^2}{2} + V(x) =& V(x_2) \\ m\dot{x}^2 =& 2\left(V(x_2) - V(x)\right) \\ \dot{x} =& \sqrt{\frac{2\left(V(x_2) - V(x)\right)}{m}} \end{aligned}$$

Now that we know the velocity at any point during the oscillation we can solve for the time it takes to get from x_1 to x_2 which is $\frac{1}{2}$ the period. Hence, the period of oscillation ,τ is found by

$$\begin{aligned} \dot{x} =& \sqrt{\frac{2\left(V(x_2) - V(x)\right)}{m}} \\ \frac{dx}{dt} =& \sqrt{\frac{2\left(V(x_2) - V(x)\right)}{m}} \\ dt =& \frac{dx}{\sqrt{\frac{2(V(x_2)-V(x))}{m}}} \\ 2\int dt = \tau =& 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2(V(x_2)-V(x))}{m}}} \\ \tau =& 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2\left(V(x_2) - V(x)\right)}}dx \end{aligned} \tag{24}$$

2.1.2 Part B

We see that the total energy of oscillator with amplitude a is

$$E = V(a) = \frac{m\omega_0^2}{2}\left(a^2 - ba^4\right))$$

Using the equation we just derived previously for period we get

$$\begin{aligned} \tau =& 2\int_{x_1}^{x_2} \sqrt{\frac{m}{2\left(V(x_2) - V(x)\right)}}dx \\ \tau =& 2\int_{-a}^{a} \sqrt{\frac{m}{2\left(V(a) - V(x)\right)}}dx \\ \tau =& 2\int_{-a}^{a} \sqrt{\frac{m}{2\left(\frac{m\omega_0^2}{2}\left(a^2 - ba^4\right) - \frac{m\omega_0^2}{2}\left(x^2 - bx^4\right)\right)}}dx \\ \tau =& 2\int_{-a}^{a} \frac{dx}{\sqrt{\left(\omega_0^2\left(a^2 - ba^4\right) - \omega_0^2\left(x^2 - bx^4\right)\right)}} \\ \tau =& \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{\left(a^2 - ba^4 - x^2 + bx^4\right)}} \\ \tau =& \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}\sqrt{1 - b\left(a^2 + x^2\right)}} \end{aligned} \tag{25}$$

2.1.3 Part C

We will use the substitution $x = asin\theta$ and binomial expansion around b. Note $dx = acos\theta d\theta$

$$\begin{aligned}
\tau =& \frac{2}{\omega_0}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}\sqrt{1-b\left(a^2+x^2\right)}}\\
\tau =& \frac{2}{\omega_0}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-a^2sin^2\theta}\sqrt{1-b\left(a^2+a^2sin^2\theta\right)}}\\
\tau =& \frac{2}{\omega_0}\int_{\theta(-a)}^{\theta(a)}\frac{dx}{acos\theta}\sqrt{1-ba^2\left(1+sin^2\theta\right)}\\
\tau =& \frac{2}{\omega_0}\int_{\theta(-a)}^{\theta(a)}\frac{acos\theta d\theta}{acos\theta}\sqrt{1-ba^2\left(1+sin^2\theta\right)}\\
\tau =& \frac{2}{\omega_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-ba^2\left(1+sin^2\theta\right)}}\\
\tau \approx& \frac{2}{\omega_0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\left(1-\frac{1}{2}ba^2\left(1+sin^2\theta\right)\right)\\
\tau \approx& \frac{2}{\omega_0}\left[\theta-\frac{\theta}{2}ba^2+\left(\frac{\theta}{2}-cos\theta sin\theta\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
\tau \approx& \frac{2\pi}{\omega_0}\left(1+\frac{3ba^2}{4}\right)
\end{aligned} \tag{26}$$

2.2 Problem 3.7

Find the equation for the trajectory of a projectile launched with velocity v at an angle α to the horizontal, assuming negligible atmospheric resistance. Given that the ground slopes at an angle β, show that the range of the projectile (measured horizontally) is

$$x = \frac{2v^2}{g}\frac{sin(\alpha-\beta)cos\alpha}{cos\beta}$$

At what angle should the projectile be launched to achieve the maximum range?

2.2.1 Part A

To solve we first write down equations for the x and z directions:

X-Direction

$$\begin{aligned}
x =& x_0 + v_{0x}t + \frac{at^2}{2}\\
x =& v_0tcos\alpha\\
\frac{x}{v_0cos\alpha} =& t
\end{aligned}$$

Z-Direction

$$\begin{aligned}
y =& y_0 + v_{0y}t + \frac{at^2}{2}\\
y =& v_0tsin\alpha - \frac{gt^2}{2}
\end{aligned}$$

Note that $y = xtan\beta$. Now using the the equation for time found with the x direction equations of motion with the y position function we get

$$\begin{aligned}
y = xtan\beta =& v_0sin\alpha t - \frac{gt^2}{2}\\
xtan\beta =& v_0\frac{x}{v_0cos\alpha}sin\alpha - \frac{g\left(\frac{x}{v_0cos\alpha}\right)^2}{2}\\
xtan\beta - \frac{x}{cos\alpha}sin\alpha =& -\frac{g\left(\frac{x}{v_0cos\alpha}\right)^2}{2}\\
xtan\beta - xtan\alpha =& \frac{gx^2}{2v_0^2cos^2\alpha}\\
x\left[tan\alpha - tan\beta\right] =& \frac{gx^2}{2v_0^2cos^2\alpha}
\end{aligned}$$

Where here we use the trigonometric identity

$$tanx - tany = secxsecysin(x - y)$$

And insert this back into the previous step.

$$\begin{aligned}
\frac{gx^2}{2v_0^2cos^2\alpha} =& [tan\alpha - tan\beta] \\
\frac{gx}{2v_0^2cos^2\alpha} =& \left[\frac{sin(\alpha - \beta)}{cos\alpha cos\beta}\right] \\
x =& \frac{2v_0^2cos^2\alpha}{g}\left[\frac{sin(\alpha - \beta)}{cos\alpha cos\beta}\right] \\
x =& \frac{2v_0^2}{g}\left[\frac{sin(\alpha - \beta)cos\alpha}{cos\beta}\right] \qquad (27)
\end{aligned}$$

2.2.2 Part B

To determine the maximum range range we must take the derivative of the range function with respect to the parameter α, then set it equal to zero.

$$\begin{aligned}
\frac{dx}{d\alpha} = 0 =& \frac{2v_0^2}{gcos\beta}\left[cos(\alpha - \beta)cos\alpha - sin(\alpha - \beta)sin\alpha\right] \\
0 =& cos(\alpha - \beta)cos\alpha - sin(\alpha - \beta)sin\alpha \\
cos(\alpha - \beta)cos\alpha =& sin(\alpha - \beta)sin\alpha \\
1 =& tan(\alpha - \beta)tan(\alpha) \\
1 =& cos(2\alpha - \beta) \\
cos^{-1}(1) =& 2\alpha - \beta \\
\pi =& 2\alpha - \beta \\
\alpha =& \frac{\pi}{2} + \frac{\beta}{2} \qquad (28)
\end{aligned}$$

2.3 Problem 3.9

Show that in the limit of strong damping (large) the time of flight of a projectile (on level ground) is approximately $\tau \approx (\omega/g + 1/\gamma)(1 - e^{-1-\gamma\omega/g})$. Show that to the same order of accuracy the range is $x \approx (u/\gamma)(1 - e^{-1-\gamma\omega/g})$. For a projectile launched at $800ms^{-1}$ with $\gamma = 0.1s^{-1}$, estimate the range for launch angles of $30°$, $20°$ and $10°$.

2.3.1 Part A

To solve we first write down Newtons Law for the x and z directions and then solve the differential equation:

X-Direction

$$\begin{aligned}
F_x = ma_x =& -\lambda v_x \\
m\ddot{x} =& -\lambda\dot{x}
\end{aligned}$$

This has solution

$$x(t) = \frac{u}{\gamma}\left(1 - e^{-\gamma t}\right)$$

Z-Direction

$$\begin{aligned}
F_y = ma_y =& -\lambda v_y - mg \\
\ddot{z} =& -\gamma\dot{z} - g
\end{aligned}$$

This has solution

$$z(t) = \left(\frac{w}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma}$$

From here we see that the particle will be in flight until it's z-position is zero again. Therefore we set the z-equation of motion equal to zero and solve for t.

$$
\begin{aligned}
z(t) &= \left(\frac{w}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) - \frac{gt}{\gamma} \\
\frac{gt}{\gamma} &= \left(\frac{w}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) \\
t &= \frac{\gamma}{g}\left(\frac{w}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - e^{-\gamma t}\right) \\
t &\approx \frac{\gamma}{g}\left(\frac{w}{\gamma} + \frac{g}{\gamma^2}\right)\left(1 - \gamma t + \frac{\gamma^2 t^2}{2}\cdots\right) \\
t &\approx \left(\frac{w}{g} + \frac{1}{\gamma}\right)\left(1 - (1 - \gamma t + \frac{\gamma^2 t^2}{2} - \frac{\gamma^3 t^3}{6} + \cdots\right) \\
t &\approx \left(\frac{w}{g} + \frac{1}{\gamma}\right)\left(\gamma t - \frac{\gamma^2 t^2}{2} + \frac{\gamma^3 t^3}{6} - \cdots\right) \\
1 &\approx \left(\frac{w\gamma}{g} + 1\right) ln\left[1 - \gamma t\right] \\
\frac{1}{\left(\frac{w\gamma}{g} + 1\right)} &\approx ln\left[1 - \gamma t\right] \\
e^{\frac{1}{\left(\frac{w\gamma}{g}+1\right)}} &\approx 1 - \gamma t \\
\gamma t &\approx 1 - e^{-1-\frac{w\gamma}{g}} \\
t &\approx \frac{1}{\gamma}\left(1 - e^{-1-\frac{w\gamma}{g}}\right)
\end{aligned}
$$

Since there was not analytic expression for t we had to use approximations such as

$$
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n}
$$

and

$$
ln|1 - x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=0}^{\infty} \frac{x^n}{n}
$$

2.3.2 Part B

Using the result from part A and the solution to x-direction equation of motion we find that the range is just

$$
\begin{aligned}
x &= \frac{u}{\gamma}\left(1 - e^{-\gamma t}\right) \\
x &\approx \frac{u}{\gamma}\left(1 - (1 - \gamma t + \frac{\gamma^2 t^2}{2} - \frac{\gamma^3 t^3}{6} + \cdots\right) \\
x &\approx \frac{u}{\gamma}\left(\gamma t + \cdots\right)
\end{aligned}
$$

Now we use the time time we just found in Part A into this first order approximation

$$
\begin{aligned}
x &\approx \frac{u}{\gamma}\left(\gamma t\right) \\
x &\approx \frac{u}{\gamma}\left(\gamma\right) \\
x &\approx \frac{u}{\gamma}\gamma\frac{1}{\gamma}\left(1 - e^{-1-\frac{w\gamma}{g}}\right) \\
x &\approx \frac{u}{\gamma}\left(1 - e^{-1-\frac{w\gamma}{g}}\right)
\end{aligned}
\tag{29}
$$

2.3.3 Part C

For $\gamma = 0.1s^{-1}$ and $u = 800ms^{-1}cos(\alpha)$ and $w = 800ms^{-1}sin(\alpha)$ where α is the launch angle. The ranges for the corresponding launch angles $30°$, $20°$, and $10°$ are as follows

$$x_{30^\circ} = 6928m \qquad x_{20^\circ} = 7518m \qquad x_{10^\circ} = 7879m$$

Where I have just used the range equation found in part be with numerical values that given. When plugged in the $1 - e^{-1-\frac{w\gamma}{g}} \approx 1$ hence the range was determined purely by the limiting velocity value

$$Range \approx \frac{u}{\gamma} = \frac{800ms^{-1}cos\alpha}{0.1ms^{-1}}$$