Physics 105 Chapter 1

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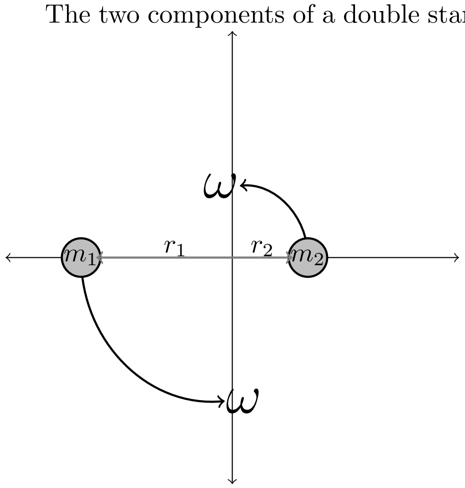
1 Homework 1 Problems

1.1 Problem 1.2

The two components of a double star are observed to move in circles of radii r_1 and r_2. What is the ratio of their masses?



We know that both stars exert equal and opposite forces of each other, $F_{12} = -F_{21}$. This is the only force that could cause angular acceleration. Both stars feel the same angular acceleration. The stars separation is given by $\Delta\vec{r} = \vec{r_1} - \vec{r_2}$ and the force on each star is given by

$$F_{12} = -F_{21} = G\frac{Mm}{\Delta\vec{r}}$$

We apply Newtons second law to obtain

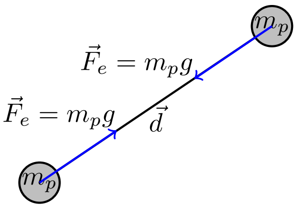
$$F_{12} = ma_{c_1} = m\omega^2 r_1 \qquad F_{21} = Ma_{c_2} = M\omega^2 r_2$$

Finally we take the ratios of the forces an solve for the ratio of radii

$$\begin{aligned} \frac{F_{12}}{F_{21}} &= \frac{m_1 a_{c_1}}{m_2 a_{c_2}} \\ \frac{G\frac{m_1 m_2}{\Delta\vec{r}}}{G\frac{m_2 m_1}{\Delta\vec{r}}} &= \frac{m_1\omega^2 r_1}{m_2\omega^2 r_2} \\ 1 &= \frac{m_1 r_1}{m_2 r_2} \\ \frac{r_2}{r_1} &= \frac{m_1}{m_2} \qquad (1) \end{aligned}$$

1.2 Problem 1.4

Find the distance r between two protons at which the electrostatic repulsion between them will equal the gravitational attraction of the Earth on one of them. ($m_p = 1.7 \times 10^{-27} kg$ and $e^+ = 1.6 \times^{-19} C$.)

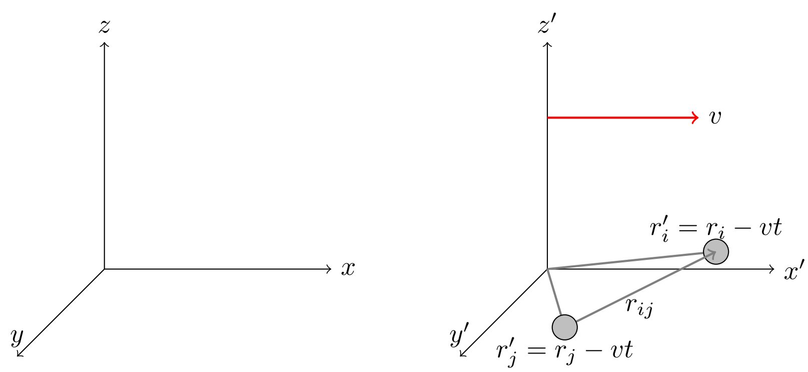


First we find the force of gravity on the proton, which is just $F_g = m_p g$. Then we set it equal to the Coulomb electrostatic equation

$$\begin{aligned} F_g &= F_e \\ m_p g &= k\frac{e^{2+}}{d^2} \\ d^2 &= k\frac{e^{2+}}{m_p g} \\ d &= \pm\sqrt{\frac{ke^{2+}}{m_p g}} \\ d &= \pm\sqrt{\frac{9\times 10^9 \frac{N\cdot m^2}{C^2}(1.6\times 10^{-19}C)^2}{1.7\times 10^{-27}kg 9.8\frac{m}{s^2}}} \\ d &= 2.94\times 10^8 m \end{aligned}$$

1.3 Problem 1.5

Consider a transformation to a relatively uniformly moving frame of reference, where each position vector r_i is replaced by $r_i' = r_i vt$. (Here v is a constant, the relative velocity of the two frames.) How does a relative position vector r_{ij} transform? How do momenta and forces transform? Show explicitly that if equations (1.1) to (1.4) hold in the original frame, then they also hold in the new one.



1.3.1 Part A

We can see that the relative position is unchanged when transforming from either reference frame. If $r_i' = r_i - vt$ then the position of a second object, j, is given by $r_j' = r_j - vt$. Rearranging these equations to get the relative positions in terms of the stationary frame, $r_i = r_i' + vt$ and $r_j = r_j' + vt$. Now the relative position in the old frame is $r_{ij} = r_i - r_j$. If we replace these old coordinates with the new frames coordinates through the Galilean transformation we obtain

$$\begin{aligned} r_{ij} =& r_i - r_j \\ r_{ij} =& (r_i' + vt) - (r_j' + vt) \\ r_{ij} =& r_i' + vt - r_j' - vt \\ r_{ij} =& r_i' - r_j' \\ r_{ij} =& r_{ij}' \end{aligned}$$

1.3.2 Part B

How do momenta and forces transform? Show explicitly that equations (1.1) and (1.4) hold in the original frame, then they also hold in the new frame.

Momenta is defined as mass times velocity. The i-th particles momentum is defined as $p_i = m_i v_i = m_i \frac{dr_i}{dt}$. If we take the derivative of the position vector we obtain

$$\begin{aligned}
p_i =& m_i v_i \\
=& m_i \frac{dr_i}{dt} \\
=& m_i \frac{d}{dt}\left(r'_i + vt\right) \\
=& m_i \left(\frac{dr'_i}{dt} + v\frac{dt}{dt}\right) \\
p_i =& m_i \left(\frac{dr'_i}{dt} + v\right) \\
p_i =& p'_i + m_i v
\end{aligned}$$

We see that the transformation from old reference frame to new one is

$$p'_i = p_i - m_i v$$

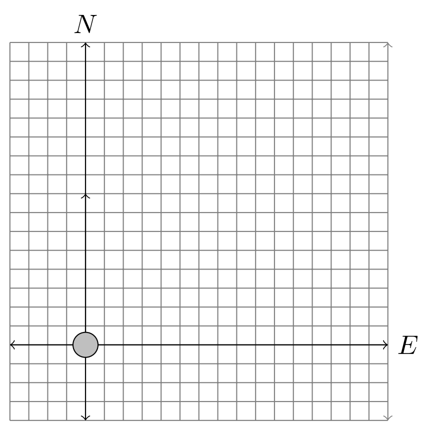
For the force transformation we can show that equation (1.1) holds for both frames. Equation (1.1) is $\dot{p} = ma = F$. If we take the time derivative with the position transformation of the momentum we find that

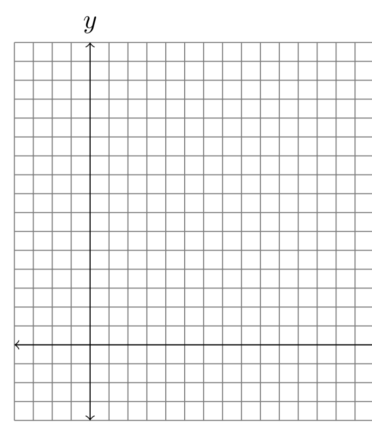
$$\begin{aligned}
\dot{p} =& m_i \frac{d}{dt}\left(\frac{dr'_i}{dt} + v\right) \\
=& m_i \left(\frac{d}{dt}\frac{dr'_i}{dt} + \frac{d}{dt}v\right) \\
=& m_i \frac{d^2 r'_i}{dt^2} \\
\dot{p} =& m a'_i
\end{aligned}$$

Since by definition $\dot{p}$ is equal to ma we can conclude that a force is equivalent in any reference frame while momentum is not. Momenta is just a constant (the relative velocity of the reference frames) times the mass.

1.4 Problem 1.7

An aircraft is to fly to a destination $800km$ due north of its starting point. Its airspeed is $800\frac{km}{hr}$. The wind is from the east at a speed of $30\frac{m}{s}$. On what compass heading should the pilot fly? How long will the flight take? If the wind speed increases to $50\frac{m}{s}$, and the wind backs to the north-east, but no allowance is made for this change, how far from its destination will the aircraft be at its expected arrival time, and in what direction?





1.4.1 Part A

I will first convert the wind speed from ms^{-1} to kmh^{-1}.

$$30\frac{m}{s} \times \frac{3600sec}{1hr} \times \frac{1km}{1000m} = 108\frac{km}{hr}$$

Having the wind and airplane velocities in same units we will just use vector addition, with $v_p = \left(0\hat{\imath}, 800\hat{\jmath}\right)\frac{km}{hr}$ and $v_w = \left(-108\hat{\imath}, 0\hat{\jmath}\right)\frac{km}{hr}$ with a combined velocity of $v_t = \left(-108\hat{\imath}, 800\hat{\jmath}\right)\frac{km}{hr}$. To get the angle with which the plane should leave we must find the angle at which the planes speed will contribute $108\frac{km}{hr}$ to the east.

$$800\frac{km}{hr} \cdot sin(\theta) = 108\frac{km}{hr}$$
$$\theta = sin^{-1}\left(\frac{108\frac{km}{hr}}{800\frac{km}{hr}}\right) = 7.8° \quad (2)$$

This answer is quoted as 7.8° east of north.

1.4.2 Part B

Since the the destination is directly north we use trigonometry to determine the y-velocity at 7.6° north of west heading. The y-velocity is then just

$$v_y = 800\frac{km}{hr}cos(7.8°) = 792\frac{km}{hr}$$

Hence, the time it takes to reach the destination is then just the distance divided by the y- velocity.

$$t = \frac{800km}{792\frac{km}{hr}} = 1.01hr = 60.6min \quad (3)$$

1.4.3 Part C

If the wind increases to $50\frac{m}{s}$ ($180\frac{km}{hr}$) from the north-east and no corrections are made the corresponding velocity vector of the wind would be $v_w = \left(127\hat{i}, 127\hat{j}\right)\frac{km}{hr}$. Adding this velocity to the planes velocity vector we obtain

$$v_w = \left(0 - 127\hat{i}, 800 - 127\hat{j}\right)\frac{km}{hr} = \left(-127\hat{i}, 673\hat{j}\right)\frac{km}{hr}$$

.

Now by multiplying each component by the expected time we will get the position

$$\vec{r} = \left(-127\hat{i}, 673\hat{j}\right)\frac{km}{hr} \times 1.1hr = \left(-139\hat{i}, 740\hat{j}\right)$$

The distance from the destination is then just the vector subtraction of the planes position from destinations position $\vec{d} = \left(-139\hat{i}, 60\hat{j}\right)$ and then use Pythagorean theorem to obtain

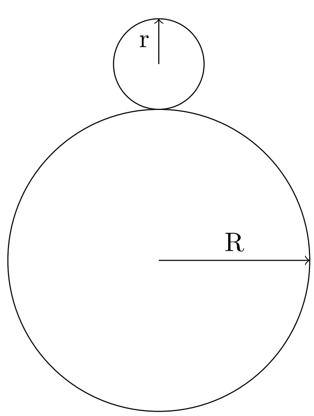
$$d = \sqrt{139^2 + 60^2} = 150km \quad (4)$$

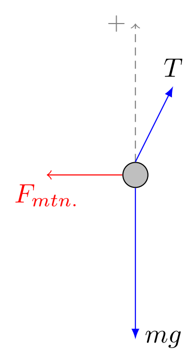
With an angle of

$$\theta = tan^{-1}\left(\frac{-127}{673}\right) = 10°\text{W of S} \quad (5)$$

1.5 Problem 1.10

The first estimate of Newtons constant was made by the astronomer Nevil Maskelyne in 1774 by measuring the angle between the directions of the apparent plumb-line vertical on opposite sides of the Scottish mountain Schiehallion (height $1081m$, chosen for its regular conical shape). Find a rough estimate of the angle through which a plumb line is deviated by the gravitational attraction of the mountain, by modelling the mountain as a sphere of radius $500m$ and density $2.7 \times 10^3\frac{kg}{m^3}$, and assuming that its gravitational effect is the same as though the total mass were concentrated at its centre.





The force due to the mountain is given by

$$F_{mtn.} = G\frac{m_{mtn}m}{r^2}$$

Where the mass of the mountain is $m_{mtn.} = V\rho = \frac{4}{3}\pi(500m)^3 \times 2.7 \times 10^3 \frac{kg}{m^3} = 1.4 \times 10^{12}kg$. Since the plumline is in static equilibrium the x-direction tension must equal the force of the mountain and the y-direction must equal the force of the earth. The force due to the mountain is $f_{mtn.} = G\frac{1.4\times10^{12}kg\cdot m}{500^2} = m \cdot 3.73 \times 10^{-4}$ and the force due to earth is mg. The angle is then found by taking the inverse tangent of these two forces

$$\theta = tan^{-1}\left(\frac{m \cdot 3.73 \times 10^{-4}}{mg}\right) = tan^{-1}\left(\frac{3.73 \times 10^{-4}}{9.8}\right) \approx 2.15 \times 10^{-3}\text{degrees} \tag{6}$$

1.6 Problem 2.6

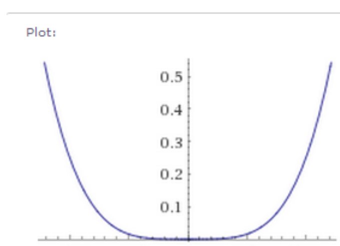
A particle of mass m moves under a force $F = cx^3$, where c is a positive constant. Find the potential energy function. If the particle starts from rest at $x = a$, what is its velocity when it reaches $x = 0$? Where in the subsequent motion does it instantaneously come to rest?

1.6.1 Part A

We use the definition of force, $F = -\frac{\partial U}{\partial x}$. To find the potential we just integrate the force function with respect to x.

$$\begin{aligned} F =& -cx^3 \\ -\frac{\partial U}{\partial x} =& -cx^3 \\ \int^{U(x)} \partial U =& -\int^{x} cx^3 \partial x \\ U(x) =& \frac{cx^4}{4} \end{aligned} \tag{7}$$

The graph of the potential function looks like as follows



1.6.2 Part B

To find the particles velocity at $x = 0$ we just use energy conservation

$$\begin{aligned} E_{x=-a} =& E_{x=0} \\ \frac{ca^4}{4} + 0 =& \frac{mv^2}{2} + 0 \\ \frac{ca^4}{2m} =& v^2 \\ v =& \pm\sqrt{\frac{c}{2m}}a^2 \end{aligned} \tag{8}$$

1.6.3 Part C

The points where the particle have zero velocity are where the potential is a maximum.

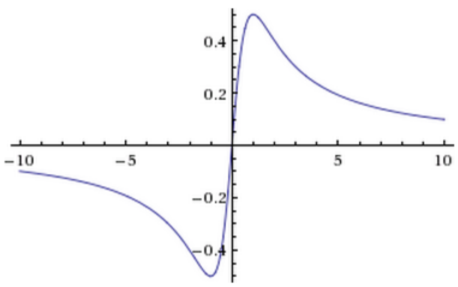
$$\begin{aligned} E =& T + V \\ E =& V \\ \frac{ca^4}{4} =& \frac{cx^4}{4} \\ x =& \pm a \end{aligned} \tag{9}$$

1.7 Problem 2.12

The potential energy function of a particle of mass m is $V = \frac{cx}{x^2+a^2}$, where c and a are positive constants. Sketch V as a function of x. Find the position of stable equilibrium, and the period of small oscillations about it. Given that the particle starts from this point with velocity v, find the ranges of values of v for which it (a) oscillates, (b) escapes to ∞, and (c) escapes to $+\infty$.

1.7.1 Part *i*

The potential energy function is $V(x) = \frac{cx}{x^2+a^2}$ hence the graph looks like



1.7.2 Part *ii*

To find stable equilibrium we need to find where the force equals zero, and where the potential function is concave up.

$$\begin{aligned}
F =& -\frac{dV}{dx} \\
-\frac{dV}{dx} =& \frac{d}{dx}\left(\frac{cx}{a^2+x^2}\right) \\
0 =& \frac{-c(x^2+a^2)+cx(2x)}{(x^2+a^2)^2} \\
0 =& \frac{-cx^2-ca^2+2cx^2}{(x^2+a^2)^2} \\
0 =& \frac{-cx^2-ca^2+2cx^2}{(x^2+a^2)^2} \\
0 =& \frac{cx^2-ca^2}{(x^2+a^2)^2} \\
0 =& \frac{cx^2}{(x^2+a^2)^2} - \frac{ca^2}{(x^2+a^2)^2} \\
\frac{cx^2}{(x^2+a^2)^2} =& \frac{ca^2}{(x^2+a^2)^2} \\
cx^2 =& ca^2 \\
x =& \pm a
\end{aligned}$$

Now we check concavity by taking second derivative of this function and plugging in $x = \pm a$. However, we can see from the graph of $V(x)$ that only $x = -a$ is a stable equilibrium point. I found the derivative using Wolfram Alpha

$$V''(x) = \frac{2cx(x^2-3a^2)}{(a^2+x^2)^3}$$

By plugging in both values we see that $V''(-a) > 0$ and $V''(-a) < 0$. Hence, stable equilibrium exists only at

$$x_{equilibrium} = -a \tag{10}$$

1.7.3 Part *iii*

To find the period of small oscillations about the equilibrium we use our definition of the period

$$
\begin{aligned}
T =& 2\pi\sqrt{\frac{m}{V''(-a)}} \\
T =& 2\pi\sqrt{\frac{m}{\frac{4ca^3}{(2a^2)^3}}} \\
T =& 2\pi\sqrt{\frac{8ma^6}{4ca^3}} \\
T =& 2\pi\sqrt{\frac{2ma^3}{c}}
\end{aligned} \tag{11}
$$

1.7.4 Part A

For the particle to oscillate it must not be able to escape to negative or positive infinity. Consequently, we must find values of v which keep the particles kinetic energy less than the potential energy.

$$
\begin{aligned}
T <& |V(-a)| \\
\frac{mv^2}{2} <& \frac{c}{2a} \\
v^2 <& \frac{c}{ma} \\
v <& \pm\sqrt{\frac{c}{ma}}
\end{aligned} \tag{12}
$$

1.7.5 Part B

To see what values of velocity the particle needs to escape to $-\infty$ we look at the limit

$$\lim_{x\to-\infty}\frac{cx}{x^2+a^2} = 0$$

The total energy of the system must be greater than or equal to zero for the particle to escape. The range of velocities, v, are found by looking at the total energy

$$
\begin{aligned}
E_i =& E_f \\
V(-a) =& \frac{mv^2}{2} \\
\frac{c}{2a} =& \frac{mv^2}{2} \\
v^2 =& \frac{c}{ma} \\
v =& \pm\sqrt{\frac{c}{ma}} \longrightarrow |v| > \sqrt{\frac{c}{ma}}
\end{aligned}
$$

However, if the velocity is larger than $\sqrt{\frac{2c}{ma}}$ it will escape to positive infinity, as we will see in the next part. Hence,

$$v < -\sqrt{\frac{c}{ma}} \quad or \quad \sqrt{\frac{c}{ma}} < v < \sqrt{\frac{2c}{ma}} \tag{13}$$

1.7.6 Part C

To see what values of velocity the particle needs to escape to ∞ we look at the limit

$$\lim_{x\to\infty}\frac{cx}{x^2+a^2} = 0$$

$$
\begin{aligned}
E_i =& E_f \\
2V(-a) =& \frac{mv^2}{2} \\
\frac{c}{a} =& \frac{mv^2}{2} \\
v^2 =& \frac{2c}{ma} \\
v =& \pm\sqrt{\frac{2c}{ma}}
\end{aligned}
$$

If the velocity is large enough to escape to positive infinity then it will also be large enough to escape to negative infinity. Therefore we must restrict our velocity to positive velocities

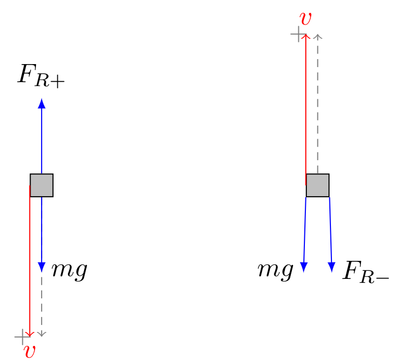
$$v > \sqrt{\frac{2c}{ma}} \tag{14}$$

1.8 Problem 2.15

A particle moves vertically under gravity and a retarding force proportional to the square of its velocity. If v is its downward or upward speed, show that $\dot{v} = \pm gkv^2$, respectively, where k is a constant. If the particle is moving upwards, show that its position at time t is given by $z = z_0 + (1/k)lncos[\sqrt{gk}(t_0 t)]$, where z_0 and t_0 are integration constants. If its initial velocity at $t = 0$ is u, find the time at which it comes instantaneously to rest, and its height then.

1.8.1 Part A

Since the motion of the particle is retarded by a force that is proportional to the square of its velocity, we will define the retardation force as $F_R = -\alpha v^2$. Looking at the forces acting on the particle we get a ODE. Here is the free body diagram for both upward and downward motion



Using the fact that $\dot{v} = a$, and that we change from up being positive to down being positive for the respective motions. We find that

$$\begin{aligned}\sum F &= ma \\ -\alpha v^2 \mp mg &= m\dot{v} \\ -\frac{\alpha}{m}v^2 \mp g &= \dot{v} \\ -kv^2 \mp g &= \dot{v}\end{aligned} \tag{15}$$

Where we define $\frac{\alpha}{m} = k$.

1.8.2 Part B

We begin by finding the acceleration function for upward motion, then integrating it. Using the FBD we find that [1]

$$
\begin{aligned}
\dot{v} =& -kv^2 - g \\
\frac{dv}{dt} =& -kv^2 - g \\
-\frac{1}{g}\int_{v_0}^{v} \frac{dv}{\frac{k}{g}v^2 + 1} =& \int_{t_0}^{t} dt \\
\frac{1}{\sqrt{gk}} tan^{-1}\left(\sqrt{\frac{k}{g}}v\right) =& t - t_0 \\
tan^{-1}\left(\sqrt{\frac{k}{g}}v\right) =& \sqrt{gk}\,(t - t_0) \\
\sqrt{\frac{k}{g}}v =& tan\left[\sqrt{gk}\,(t - t_0)\right] \\
v =& \sqrt{\frac{g}{k}} tan\left[\sqrt{gk}\,(t - t_0)\right] \qquad (16) \\
\frac{dz}{dt} =& \sqrt{\frac{g}{k}} tan\left[\sqrt{gk}\,(t - t_0)\right] \\
\int_{z_0}^{z} dz =& \int_{t_0}^{t} \sqrt{\frac{g}{k}} tan\left[\sqrt{gk}\,(t - t_0)\right] \\
z - z_0 =& \int_{t_0}^{t} \sqrt{\frac{g}{k}} tan\left[\sqrt{gk}\,(t - t_0)\right] \\
z =& z_0 + \sqrt{\frac{g}{k}} ln\left[cos\left(\sqrt{gk}(t - t_0)\right)\right] \cdot \sqrt{\frac{1}{gk}} \\
z =& z_0 + \frac{1}{k} ln\left[cos\left(\sqrt{gk}(t - t_0)\right)\right] \qquad (17)
\end{aligned}
$$

Where we used the known integrals

$$\int tan(u(t))du = -ln|cos(u(t))| + C$$

And

$$\int \frac{dt}{1 + bt^2} = tan^{-1}(\sqrt{b}t) + C$$

1.8.3 Part C

To find when the particle comes to rest we use Equation 10 with $v = 0$ and an initial velocity condition. Then we plug that time into Equation 11 to find its height.

$$
\begin{aligned}
v =& u - \sqrt{\frac{g}{k}} tan\left[\sqrt{gk}\,(t - t_0)\right] \\
\sqrt{\frac{g}{k}} tan\left[\sqrt{gk}\,(t - 0)\right] =& u \\
tan\left[\sqrt{gk}\,(t)\right] =& u\sqrt{\frac{k}{g}} \\
\sqrt{gk}\,(t) =& tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \\
t =& \frac{1}{\sqrt{gk}} tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \qquad (18)
\end{aligned}
$$

[1] To see full derivation please see attached handwritten sheets.

Now let us plug this value into Equation 11, with $z_0 = 0$.

$$\begin{aligned}
z =& z_0 + \frac{1}{k} ln\left[cos\left(\sqrt{gk}(t-0)\right)\right] \\
z =& \frac{1}{k} ln\left[cos\left(\sqrt{gk}\left\{\frac{1}{\sqrt{gk}} tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right\}\right)\right] \\
z =& \frac{1}{2k} ln\left[1 + tan^2\left(tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right)\right] \\
z =& \frac{1}{2k} ln\left[1 + \left(u\sqrt{\frac{k}{g}}\right)^2\right] \\
z =& \frac{1}{2k} ln\left[1 + \left(u^2\frac{k}{g}\right)\right] \qquad (19)
\end{aligned}$$

Where we have used the identity

$$ln(cosx) = \frac{1}{2} ln(1 + tan^2 x)$$

1.9 Problem 2.16

Show that if the particle of the previous question falls from rest its speed after a time t is given by $v = \frac{g}{k} tanh(\sqrt{gk}t)$. What is its limiting speed? How long does it take to hit the ground if dropped from height h?

1.9.1 Part A

If we wish to find the velocity function we first need to set up the acceleration function then integrate. We use the FBD from the previous problem and find that

$$\begin{aligned}
\dot{v} =& -kv^2 + g \\
\frac{dv}{dt} =& -kv^2 + g \\
\int_{v_0}^{v} \frac{dv}{-kv^2 + g} =& \int_{t_0}^{t} dt \\
\frac{1}{k}\int_{v_0}^{v} \frac{dv}{\frac{g}{k} - v^2} =& \int_{t_0}^{t} dt \\
\frac{1}{k}\sqrt{\frac{k}{g}} \cdot tanh^{-1}\left(\sqrt{\frac{k}{g}}v\right) =& t - t_0 \\
\frac{1}{\sqrt{kg}} \cdot tanh^{-1}\left(\sqrt{\frac{k}{g}}v\right) =& t - t_0 \\
tanh^{-1}\left(\sqrt{\frac{k}{g}}v\right) =& \sqrt{gk}(t - t_0) \\
\sqrt{\frac{k}{g}}v =& tanh\left(\sqrt{gk}(t - t_0)\right) \\
v =& \sqrt{\frac{g}{k}} \cdot tanh\left(\sqrt{gk}(t - t_0)\right) \qquad (20)
\end{aligned}$$

Where we have used the integration formula

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} tanh^{-1}\left(\frac{x}{a}\right)$$

1.9.2 Part B

Since hyperbolic tangent with any argument levels off at 1, we can conclude that the limiting speed is just the coefficient of the hyperbolic tangent function

$$v_{limit} = \sqrt{\frac{g}{k}} \qquad (21)$$

1.9.3 Part C

To find the time it takes to fall a distance h we must find the position as function of time. The derivation is similar to the position equation of the previous problem so I will omit the steps. However, we use the following integration formula

$$\int a \cdot tanh(bx)dx = \frac{a}{b} \cdot ln[cos(bx)] + C$$

Hence, the position equation is given by

$$\begin{aligned} z =& z_0 - \frac{1}{k} \cdot ln\left[cos\left(\sqrt{gk}(t-t_0)\right)\right] \\ 0 - h =& -\frac{1}{k} \cdot ln\left[cos\left(\sqrt{gk}(t)\right)\right] \\ hk =& ln\left[cos\left(\sqrt{gk}(t)\right)\right] \\ e^{hk} =& cos\left(\sqrt{gk}(t)\right) \\ cos^{-1}\left(e^{hk}\right) =& \sqrt{gk}(t) \\ \frac{1}{\sqrt{gk}} \cdot cos^{-1}\left(e^{hk}\right) =& t \end{aligned} \tag{22}$$

1.10 Problem 2.25

For an oscillator under a periodic force $F(t) = F_1 cos(\omega_1 t)$, calculate the power, the rate at which the force does work. Show that the average power is $P = m\gamma\omega_1^2\alpha_1^2$, and hence verify that it is equal to the average rate at which energy is dissipated against the resistive force. Show that the power P is a maximum, as a function of ω_1, at $\omega_1 = \omega_0$, and find the values of ω_1 for which it has half its maximum value.

1.10.1 Part A

We first set up our differential equation

$$m\ddot{x} + \lambda\dot{x} + kx = F_1 cos(\omega t)$$

the solution to this second order non-homogeneous differential equation is just

$$x = a_1 cos\left(\omega_1 t - \theta_1\right) + ae^{-\gamma t}cos\left(\omega t - \theta\right)$$

However, we will look at large t and ignore the transient term, hence the solution is just

$$x = a_1 cos\left(\omega_1 t - \theta_1\right)$$

Power by definition is $P = \frac{dW}{dt} = \frac{Fdx}{dt} = F \cdot v$ so all we need is the derivative of the position function multiplied by the force function to find power. The velocity is just $v(t) = -\omega_1 a_1 sin\left(\omega_1 t - \theta_1\right)$ and the power function is

$$P(t) = F_1\omega_1 a_1 cos(\omega_1 t) sin\left(\omega_1 t - \theta_1\right) \tag{23}$$

1.10.2 Part B

The average power will be found by integrating the power function of a period then dividing by the period. The period is just $T = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega_1}$. The magnitude of the force function $F_1 = \frac{2\gamma m a_1 \omega_1}{sin\theta_1}$.

$$\begin{aligned} \overline{P}(t) =& \frac{\int^T P(t)dt}{T} \\ =& \frac{\int^T F_1\omega_1 a_1 cos(\omega_1 t) sin\left(\omega_1 t - \theta_1\right) dt}{T} \\ =& F_1\omega_1 a_1 \frac{\omega_1}{2\pi}\left(-\pi\frac{sin(\theta_1)}{\omega_1}\right) \\ =& \frac{1}{2}F_1\omega_1 a_1 sin\left(\theta_1\right) \\ =& \frac{1}{2}\frac{2\gamma m a_1\omega_1}{sin\theta_1} sin\theta_1 \omega_1 a_1 \\ \overline{P}(t) =& m\gamma\omega_1^2 a_1^2 \end{aligned} \tag{24}$$

The average rate of energy dissipation produced by the resistive force is then just

$$\begin{aligned}
\overline{P}(t) =& \frac{\int^T P(t)dt}{T} \\
=& \frac{\omega_1}{2\pi}\int^{\frac{2\pi}{\omega}} \lambda v(t)dt \\
=& -\frac{\omega_1}{2\pi}\int^{\frac{2\pi}{\omega}} \lambda\omega_1 a_1 sin\left(\omega_1 t - \theta_1\right)dt \\
=& \frac{\omega_1}{2\pi}\left[\lambda\omega_1 a_1 \pi sin\left(\omega_1 t - \theta_1\right)\right]_0^{\frac{2\pi}{\omega}} \\
=& \frac{1}{2}\lambda\omega_1^2 a_1 \\
\overline{P}(t) =& m\gamma\omega_1^2 a_1^2
\end{aligned} \tag{25}$$

Where we used the definition of a_1 from Equation 2.28 from Kibble-Berkshire.

1.10.3 Part C

To show that $P(t)$ has a maximum as a function of omega we just take the derivative and set equal to zero.

$$\begin{aligned}
\frac{dP(t)}{dt} = 0 =& \frac{d}{dt}\left(F_1\omega_1 a_1 cos(\omega_1 t) sin\left(\omega_1 t - \theta_1\right)\right) \\
0 =& F_1\omega_1 a_1 t cos\left(\theta_1 - 2\omega_1 t\right) \\
0 =& cos\left(\theta_1 - 2\omega_1 t\right) \\
\frac{\pi}{2} =& \left(\theta_1 - 2\omega_1 t\right) \\
\frac{\pi}{2} + 2\omega_1 t =& \theta_1 \\
\frac{\pi}{2} + 2\omega_1 t =& \frac{2\gamma\omega_1}{\omega_1^2 - \omega_0^2}
\end{aligned} \tag{26}$$

We see that the frequency at which power is maximum is determined by θ_1 which is defined as $\theta_1 = tan^{-1}\left(\frac{2\gamma\omega_1}{\omega_1^2-\omega_0^2}\right) \approx \frac{2\gamma\omega_1}{\omega_1^2-\omega_0^2}$. Hence from here we see that the maximum occurs at $\omega_1 = \omega_0$.

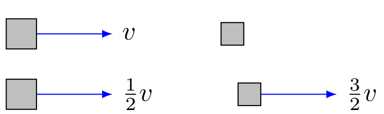
Please note that the derivatives were calculated using Wolfram.

2 Extra Credit Problems

2.1 Chapter 1 : Problem 1

An object A moving with velocity v collides with a stationary object B. After the collision, A is moving with velocity $\frac{1}{2}v$ and B is moving with velocity $\frac{3}{2}v$. (a) Find the ratio of masses. (b) If the masses stuck together what velocity would they have after colliding?

2.1.1 Part A



2.1.2 Part A

We first note that the this is a conservation of momentum problem. Since it is also an elastic collision we can say energy is also conserved. We write down the equations for conservation of momentum since it is all we need to solve.

$$\begin{aligned}
P_i &= P_f \\
m_A v &= \frac{m_A v}{2} + \frac{3m_B v}{2} \\
m_A v - \frac{m_A v}{2} &= \frac{3m_B v}{2} \\
\frac{m_A v}{2} &= \frac{3m_B v}{2} \\
\frac{m_A}{m_B} &= 3
\end{aligned} \tag{27}$$

2.1.3 Part B

We note that this is a non-elastic collision hence energy is not necessarily conserved. We write down conservation of momentum equations.

$$\begin{aligned}
P_i &= P_f \\
m_A v &= (m_A + m_B)\, v' \\
\frac{m_A v}{(m_A + m_B)} &= v' \\
\frac{m_A v}{m_B} \frac{1}{\left(\frac{m_A}{m_B} + 1\right)} &= v' \\
3\frac{v}{3+1} &= v' \\
\frac{3v}{4} &= v'
\end{aligned} \tag{28}$$

2.2 Chapter 2: Problem 7

A particle of mass m moves (in the region $x > 0$) under a force $F = kx + \frac{c}{x}$, where k and c are positive constants. Find the corresponding potential energy function. Determine the position of equilibrium, and the frequency of small oscillations about it.

2.2.1 Part A

To find the potential function we just integrate the force function with respect to position

$$\begin{aligned}
-\int^{U(x)} dU &= \int^{x} F dx \\
-U(x) &= \int^{x} kx + \frac{c}{x} dx \\
U(X) &= \frac{kx^2}{2} - c \cdot ln|x|
\end{aligned} \tag{29}$$

2.2.2 Part B

To determine the equilibrium position we must find where the force is zero.

$$\begin{aligned}
F = 0 =& kx + \frac{c}{x} \\
kx =& \frac{c}{x} \\
x^2 =& \frac{c}{k} \\
x = \pm& \sqrt{\frac{c}{k}} \longrightarrow (x > 0) \longrightarrow x = \sqrt{\frac{c}{k}}
\end{aligned} \tag{30}$$

2.2.3 Part C

To find the frequency of oscillation about this point we must compute the second derivative of the potential function, or equivalently the first derivative of the force function.

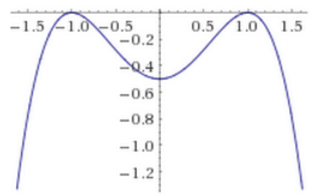
$$\begin{aligned}
-U''(x) =& \frac{d}{dx}\left(-kx + \frac{c}{x}\right) \\
=& -k - \frac{c}{x^2} \\
-U''\left(\sqrt{\frac{c}{k}}\right) =& -k - \frac{ck}{c} \\
U''\left(\sqrt{\frac{c}{k}}\right) =& 2k
\end{aligned}$$

Then we just plug into the standard frequency equation

$$f = \frac{1}{2\pi}\sqrt{\frac{U''(x)}{m}} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}} \longrightarrow \omega = \sqrt{\frac{2k}{m}} \tag{31}$$

2.3 Chapter 2: Problem 11

The potential energy function of a particle of mass m is $V = -\frac{1}{2}c(x^2 - a^2)^2$, where c and a are positive constants. Sketch this function, and describe the possible types of motion in the three cases (a) $E > 0$, (b) $E < -\frac{1}{2}ca^4$, and (c) $-\frac{1}{2}ca^4 < E < 0$.



At $x = 0$ the potential energy is $V(0) = -\frac{1}{2}ca^4$. and at the peaks the potential is zero.

2.3.1 Part A

For $E > 0$ the particle will simply move through this potential. It will slow down where the potential is max and speed up where potential is minimum. No turning points.

2.3.2 Part B

For a particle with $E < -\frac{1}{2}ca^4$ it will not have enough energy to get into the center well of the potential function, so it will have one turning point.

2.3.3 Part C

For a particle with $-\frac{1}{2}ca^4 < E < 0$ the particle it could not make it into the center potential well or just get to the peaks of it and depending on certain conditions turn around at the first or second peak, make it over one, or make it over both. It can have up to two turning points.