Physics 105 Homework 3

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1 Problem 4.12

1.1 Part A

As the planet moves an infinitesimal distance through the cloud it will pick up an infinitesimal amount of mass from the particles. The infinitesimal distance it travels is given by $dx = vdt$. This infinitesimal distance can be translated to a volume of the particles it picks up. The relation between mass and volume eith constant density is $dM = \rho dV$ putting this together we get.

$$dV = \pi r^2 dx = \pi r^2 v dt \longrightarrow dV = \frac{dM}{dt} = \pi r^2 v dt \longrightarrow \frac{dM}{dt} = \rho \pi r^2 v$$

Now we know that the particles, that are originally outside the stars radius, will be attracted to within the stars radius. We need to determine the radius of a circle of the cloud (when at a very far distance away) that will be hit the star. To do this assume all the mass of the star is at the center of it. Then we just need the impact parameter that will bring the particles to within the stars radius. To do so we use the distance of closest approach equation $r_m = -a + \sqrt{a^2 + b^2}$ where $a = \frac{GM}{v^2}$ and b is the impact parameter. Solving for the impact parameter we obtain

$$r_m = -a + \sqrt{a^2 + b^2} \longrightarrow b^2 = r_m^2 + 2ar_m$$

Since we want the distance of closest approach to be R we can plug in to obtain

$$b^2 = R^2 + \frac{2GM}{v^2}$$

This is the square of the radius of the circle of particles that will collide with the earth. Therefore,by plugging this value into our r^2 we obtain

$$\begin{aligned} \frac{dM}{dt} =& \rho \pi r^2 v \\ \frac{dM}{dt} =& \rho \pi b^2 v \\ \frac{dM}{dt} =& \rho \pi v \left(R^2 + \frac{2GM}{v^2} \right) \end{aligned} \tag{1}$$

1.2 Part B

A star has mass $M = 10^{31} kg$ and radius $R = 10^8 km = 10^1 1m$. Since particles of the cloud are attracted to the stars center more cloud particles would hit the star than if there were no gravitational effects. Therefore the effective cross sectional area is πb^2 and the geometric cross sectional area is πR^2. The ratio of these two is

$$\frac{\sigma_{eff}}{\sigma_{geo}} = \frac{\pi b^2}{\pi R^2} = \frac{R^2 + \frac{2GMR}{v^2}}{R^2} = 1 + \frac{2GM}{v^2 R}$$

1.2.1 Case 1: $v = 1000\frac{km}{s}$

$$\frac{\sigma_{eff}}{\sigma_{geo}} = 1 + \frac{2GM}{v^2 R} = 1 + \frac{2G \cdot 10^{31} kg}{(1000\frac{km}{s})^2 \cdot 10^8 km} = 1 + 0.01334 = 1.01334 \tag{2}$$

1.2.2 Case 2: $v = 100\frac{km}{s}$

$$\frac{\sigma_{eff}}{\sigma_{geo}} = 1 + \frac{2GM}{v^2 R} = 1 + \frac{2G \cdot 10^{31} kg}{(100\frac{km}{s})^2 \cdot 10^8 km} = 1 + 1.334 = 2.334 \tag{3}$$

1.2.3 Case 2: $v = 100\frac{km}{s}$

$$\frac{\sigma_{eff}}{\sigma_{geo}} = 1 + \frac{2GM}{v^2 R} = 1 + \frac{2G \cdot 10^{31} kg}{(10\frac{km}{s})^2 \cdot 10^8 km} = 1 + 133.4 = 134.4 \tag{4}$$

2 Problem 4.16

2.1 Part A

We know that the speed of the spaceship at perihelion is $38.6\frac{km}{s}$ and the speed of earth is $29.8\frac{km}{s}$. All we need to do to find the relative velocity of the spaceship with respect to earth is subtract the two velocities.

$$v_{rel} = v_{ss} - v_e = 38.6\frac{km}{s} - 29.6\frac{km}{s} = 8.8\frac{km}{s} \tag{5}$$

2.2 Part B

We know that the speed of the spaceship at aphelion is $7.4\frac{km}{s}$ and the speed of jupiter is $13.1\frac{km}{s}$. All we need to do to find the relative velocity of the spaceship with respect to jupiter is subtract the two velocities.

$$v_{rel} = v_{ss} - v_j = 7.4\frac{km}{s} - 13.1\frac{km}{s} = -5.7\frac{km}{s} \tag{6}$$

Since the spaceship is slower than jupiter at that point we assume the spaceship is just slightly in front of jupiter and is moving in the negative direction (getting closer to jupiter) with respect to jupiter.

2.3 Part C

Since the spaceships orbit just touches earths and jupters orbit we can assume the spaceship leaves earth parallel to its velocity and will arrive at jupiter after half of its full orbit. Using Kepler's 3rd law we can assume that the time it takes to travel from earth to jupiter will be half of the period. Now we must determine how far jupiter travels in one half of the spaceships orbit.

The period,τ, of the spaceships orbit is 5.46 years, and a half of its orbit is 2.73 years. By using the arc length relationship we can find the angle at which jupiter must be from the earth sun axis to meet the spaceship after 2.73 years.

$$\alpha = \frac{s}{r} = \frac{v_j \tau}{2 \cdot R_j} = \frac{13.1\frac{km}{s} \cdot 5.46 yrs}{2 \cdot 5.2 AU} = 82.8^\circ \text{ From the } +\text{x}-\text{axis} \longrightarrow 180^\circ - 82.8^\circ = 97.2^\circ \text{Ahead of Earth} \tag{7}$$

2.4 Part D

To find the angle between earth and jupiter when the spaceship arrives to jupiter we need to see how far from earths starting point is earth when the spaceship reaches jupiter. Since it takes 2.73 years for the spaceship to travel to jupiter the earth will travel for 2.73 years, and hence 2.73 rotations.

$$\alpha = \frac{s}{r} = \frac{v_e \tau}{2 \cdot R_e} = \frac{29.8\frac{km}{s} \cdot 5.46 yrs}{2 \cdot 1 AU} = 981.5^\circ \text{ From the } +\text{x}-\text{axis} \longrightarrow 900^\circ - 981.5^\circ = 81.5^\circ \text{Ahead of Jupiter} \tag{8}$$

Although earth is 82 degrees ahead of jupiter it has actually gone two full rotations before the spaceship even arrived at jupiter.

3 Problem 4.21

3.1 Part A

Just before scattering the spaceship is traveling at its slowest speed, $7.4\frac{km}{s}$, (given in problem 5) and jupiters speed is a constant $13.1\frac{km}{s}$. The relative velocity of the spaceship before is scattering is then

$$v_{rel} = v_{ss} - v_j = 7.4\frac{km}{s} - 13.1\frac{km}{s} = -5.7\frac{km}{s} \tag{9}$$

Where negative implies opposite of jupiters motion.

3.2 Part B

Just before scattering the spaceship is traveling at its slowest speed, $7.4\frac{km}{s}$, (given in problem 5) and jupiters speed is a constant $13.1\frac{km}{s}$. The relative velocity of the spaceship before is scattering is then

$$v_{rel} = v_j - v_{ss} = 13.1\frac{km}{s} - 7.4\frac{km}{s} = 5.7\frac{km}{s} \tag{10}$$

3.3 Part C

To calculate the impact parameter we can use energy conservation, for a $\frac{|k|}{r}$ potential, and a description of hyperbolic orbits (derivation omitted) to obtain the equation for the impact parameter. Note $|k|= GMm$ in this case, and the scattering angle is $\Theta = \frac{\pi}{2}$.

$$\begin{aligned}
b =& \frac{|k|}{mv^2} cot^2\left(\frac{\Theta}{2}\right) \\
b =& \frac{GMm}{mv^2} cot^2\left(\frac{\Theta}{2}\right) \\
b =& \frac{GM}{v^2} cot^2\left(\frac{\pi}{4}\right) \\
b =& \frac{GM}{v^2} cot^2\left(\frac{\pi}{4}\right) \\
b =& \frac{G \cdot 318 \cdot 5.94 \times 10^{24} kg}{(5.7\frac{km}{s})^2} \\
b =& 3.87 \times 10^6 km \approx 3.9 \times 10^6 km
\end{aligned} \tag{11}$$

3.4 Part D

Using the radial energy equation

$$\frac{mv^2}{2} = -\frac{GMm}{r_m} + \frac{J}{2mr_m^2} = -\frac{GMm}{r_m} + \frac{mvb}{2mr_m^2}$$

We can solve for the distance of closest approach,r_m, by using the quadratic formula we obtain $r_m = -a + \sqrt{a^2 + b^2}$. Where $a = \frac{GM}{v^2}$ and b is the impact parameter we calculated in the previous section.

$$\begin{aligned}
r_m =& -a + \sqrt{a^2 + b^2} \\
r_m =& \frac{GM}{v^2} + \sqrt{\frac{G^2 M^2}{v^4} + b^2} \\
r_m =& \frac{G \cdot 318 \cdot 5.94 \times 10^{24} kg}{(5.7\frac{km}{s})^2} + \sqrt{\left(\frac{G \cdot 318 \cdot 5.94 \times 10^{24} kg}{(5.7\frac{km}{s})^2}\right)^2 + (3.87 \times 10^6 km)^2} \\
r_m =& 1,600,713,526 km = 22.7 R_J \approx 23 R_J
\end{aligned} \tag{12}$$

4 Problem 4.22

4.1 Part A

If we put ourselves in jupiters reference frame we will see the slingshot affect of the spaceship as just a changing of direction to radially away from sun. However it will also pick up a tangential (to the jupiter-sun axis) component of velocity equivalent to that of jupiters. Hence, the resulting velocity of the spaceship is just found by vector addition

$$v_{sun-ss} = \sqrt{v_{ss}^2 + v_j^2} = \sqrt{5.7\tfrac{km}{s}^2 + 13.1\tfrac{km}{s}^2} = 14.3\tfrac{km}{s} \tag{13}$$

To find the direction of the resulting velocity relative to the sun we can use trigonometry

$$\theta = tan^{-1}\left(\frac{v_j}{v_{ss}}\right) = tan^{-1}\left(\frac{5.7\frac{km}{s}}{13.1\frac{km}{s}}\right) = 23.5^\circ \tag{14}$$

4.2 Part B

The act of the slingshot around jupiter applied an external torque. Just after slingshot maneuver the new angular momentum of the spaceship is just the tangential velocity component of its new velocity (jupiters velocity) times the jupiter-sun distance (5.2AU). Now we can use the radial energy equation to determine the aphelion distance. The initial energy is just

$$\begin{aligned}
E_i =& \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mR_J^2} + V(R_J) \\
E_i =& \frac{1}{2}mv_s^2 + \frac{m^2v_J^2R_J^2}{2mR_J^2} - G\frac{Mm}{R_J} \\
E_i =& m\left(\frac{v_s^2}{2} + \frac{v_J^2}{2} - G\frac{M}{R_J}\right) \\
E_i =& \frac{m}{2}\left((5.7\tfrac{km}{s})^2 + (13.1\tfrac{km}{s})^2 - 2G\frac{1.99\times10^{30}kg}{5.2\cdot1.5\times10^{11}m}\right) \\
E_i =& -6.812\times10^7 J
\end{aligned}$$

We know that when the spaceship is at the aphelion its radial velocity ($\dot{r} = 0$)is zero. The energy at the aphelion is then

$$\begin{aligned}
E_f =& \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mR_a^2} + V(R_a) \\
E_f =& \frac{m^2v_J^2R_J^2}{2mR_a^2} - G\frac{Mm}{R_a} \\
E_f =& \frac{mv_J^2R_J^2}{2R_a^2} - G\frac{Mm}{R_a}
\end{aligned}$$

Now we can equate this to the energy at the aphelion.

$$\begin{aligned}
E_i =& E_f \\
E_i =& \frac{mv_J^2R_J^2}{2R_a^2} - G\frac{Mm}{R_a} \\
E_iR_a^2 =& \frac{mv_J^2R_J^2}{2} - GMmR_a \\
2E_iR_a^2 + 2GMR_a - v_J^2R_J^2 =& 0
\end{aligned}$$

Where there is a common factor, m, that is divided out on both sides. By solving this quadratic with the obtain

$$R_a = \frac{-GM \pm \sqrt{G^2M^2 - 4(2E_i)(-v_J^2 R_J^2)}}{2(2E_i)} = 1.39 \times 10^{12} m \approx 9.26 AU \tag{15}$$

Please note that here we took the larger root. The smaller root is actually the perihelion.

4.3 Part C

To find the perihelion we take the smaller root of the quadratic equation obtained in the previous section. Doing so we obtain

$$R_p = \frac{-GM \pm \sqrt{G^2M^2 - 4(2E_i)(-v_J^2 R_J^2)}}{2(2E_i)} = 5.4 \times 10^{11} m \approx 3.6 AU \tag{16}$$

4.4 Part D

To find the period we will use the standard period equation. We will first need to calculate the semi-major axis. The semi- major axis is just the average of the perihelion distance and the aphelion distance.

$$a = \frac{3.6AU + 9.26AU}{2} = 6.43AU = 9.645 \times 10^{11} m$$

Plugging into the period equation derived from Kepler's Law we obtain

$$\tau = 2\pi\sqrt{\frac{a^3}{GM}} = 2\pi\sqrt{\frac{(9.645 \times 10^{11} m)^3}{G \cdot 1.99 \times 10^{30} kg}} = 16.38\text{years} \tag{17}$$

5 Problem 4.28

5.1 Part A

We know that the amount of flux of particles passing through the wall is given by an exponential decay. The rate of decay is determined by the mean free path, which is defined as $\lambda = \frac{1}{N\sigma}$, where N is number of particles per unit volume and σ is the cross sectional area of the atoms. For the given situation our mean free path is $\lambda = \frac{1}{2\times 10^{29} \cdot \pi 9 \times 10^{-30} m^2} = .177m$. Now using our flux decay relation we obtain

$$\begin{aligned} f &= f_0 e^{-\frac{x}{\lambda}} \\ \frac{f}{f_0} &= e^{-\frac{x}{\lambda}} \\ \frac{1}{2} &= e^{-\frac{x}{\lambda}} \\ ln\left(\frac{1}{2}\right) &= -\frac{x}{\lambda} \\ \lambda ln(2) &= x \\ 0.177 ln(2) &= x = 0.1225m \approx 0.123m \end{aligned} \tag{18}$$

5.2 Part B

We use the same flux equation as in the previous section except with a ratio of 10^{-6}. Hence,

$$x = -0.177 ln(10^{-6}) = 0.177 ln(10^6) = 2.44m \tag{19}$$

6 Problem 4.29

6.1 Part A

We will first find the velocity of the incoming alpha particle

$$4keV = \frac{mv^2}{2} \longrightarrow v = \sqrt{\frac{8000eV \cdot 1.6 \times 10^{-19} J/eV}{4 \cdot 1.66 \times 10^{-27} kg}} = 4.39 \times 10^5 \frac{m}{s}$$

The angle of scattering, Θ, is $90°$. The impact parameter, b, is found by the following relation

$$b = \frac{kQq}{mv^2} cot^2 \left(\frac{\Theta}{2}\right) = \frac{9 \times 10^9 (13e)(2e)}{4 \cdot 1.66 \times 10^{-27} kg \cdot (4.39 \times 10^6 \frac{m}{s})^2} cot^2 \frac{\pi}{4} = 4.68 \times 10^{-12} m$$

Using the radial energy equation we can find the distance of closest approach.

$$\frac{mv^2}{2} = k\frac{Qq}{r_m} + \frac{J^2}{2mr_m^2} \longrightarrow r_m = a^2 + \sqrt{a^2 + b^2}$$

Where $a = \frac{kQq}{mv^2}$. Plugging these numbers in we obtain

$$r_m = \frac{kQq}{mv^2} + \sqrt{\left(\frac{kQq}{mv^2}\right)^2 + \left(\frac{kQq}{mv^2} cot^2\right)^2} = \frac{kQq}{mv^2}\left(1 + \sqrt{2}\right) = 1.13 \times 10^{-11} m \qquad (20)$$

6.2 Part B

To determine the rate at which particles hit the detector we must first calculate the range of angles particles can be deflected and still be detected. If the detector has area $400mm^2 = 0.0004m^2$. We need to first calculate the cross sectional area which is given by

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4sin^4\left(\frac{\theta}{2}\right)} = a^2 = b^2 = 2.19 \times 10^{-23} m^2$$

Using formula (4.48) the rate of particles being detected at the detector will be

$$\begin{aligned} \int^w dw =& Nf\frac{d\sigma}{d\Omega}\frac{1}{L^2}\int_A dA \\ w =& \frac{0.05kg \cdot 6.022 \times 10^{23}}{27} \cdot 3 \times 10^8 \frac{1}{m^2 s} \cdot 2.19 \times 10^{-23} m^2 \frac{1}{0.6^2 m^2} \cdot 0.004m^2 \\ w =& 8140\frac{1}{s} \end{aligned} \qquad (21)$$

7 Problem 7.4

7.1 Solution

First we will denote all center of momentum quantities as uppercase letters and lab quantities in lower case. The kinetic energy and potential energy of the system are

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{z}_1^2 + \frac{1}{2}\mu\dot{z}_2^2$$
$$V = MgR + \frac{1}{2}k\left(|z_1 - z_2|\right)^2$$

Now we just need to write down the Euler-Lagrange equations and solve them. The Lagrangian is

$$L = T - V = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{z}_1^2 + \frac{1}{2}\mu\dot{z}_2^2 - MgR - \frac{1}{2}k\left(z_1 - z_2\right)^2$$

The respective derivatives are

$$\frac{d}{dt}\frac{\partial L}{d\dot{R}} = M\ddot{R} \qquad \frac{\partial L}{dR} = Mg$$
$$\frac{d}{dt}\frac{\partial L}{d\dot{z}_1} = \mu\ddot{z}_1 \qquad \frac{\partial L}{dz_1} = k\left(z_1 - z_2\right)$$
$$\frac{d}{dt}\frac{\partial L}{d\dot{z}_2} = \mu\ddot{z}_2 \qquad \frac{\partial L}{dz_2} = -k\left(z_1 - z_2\right)$$

Inserting into the Euler-Lagrangian equations we obtain

$$\begin{aligned} M\ddot{R} =& - Mg \\ \ddot{R} =& g \\ \iint \ddot{R}dR =& \iint^t gdt \\ R(t) =& R_0 + Vt - \frac{1}{2}gt^2 \end{aligned}$$

Now we solve the for the individual positions within the CM frame. We see that this leads to harmonic oscillator differential equation in which we know the solution.

$$\begin{aligned} \mu\ddot{z}_1 + k\left(z_1 - z_2\right) =& 0 \\ \ddot{z}_1 - \omega^2\left(z_1 - z_2\right) =& 0 \\ z_1(t) =& c_1 + A_1 sin(\omega t) \end{aligned} \qquad \begin{aligned} \mu\ddot{z}_2 + k\left(z_1 - z_2\right) =& 0 \\ \ddot{z}_2 + \omega^2\left(z_1 - z_2\right) =& 0 \\ z_2(t) =& c_2 + A_2 sin(\omega t) \end{aligned}$$

Now using our initial conditions we can solve for the constants of integration. From equations derived in the *Kibble and Berkshire* textbook we can determine the initial velocities and positions in both the lab and CM frame.

- $\dot{z}_1 = v - \dot{r}_1^* = \frac{m_2}{M}v$
- $z_1 = \frac{\ell}{2}$
- $\dot{z}_2 = v - \dot{r}_2^* = \frac{m_1}{M}v$
- $z_2 = -\frac{\ell}{2}$
- $\dot{R} = V = \frac{m_1 v}{M}$
- $R_0 = \frac{\ell}{2}$

Now plugging in initial values at $t = 0$ for both the position function and the velocity function we obtain.

$$\begin{aligned} z_1(t) =& \frac{\ell}{2} + A_1 sin(\omega t) \\ \dot{z}_1(t) =& \omega A_1 cos(\omega t) \\ \dot{z}_1(t=0) = \frac{vm_1}{M} =& A_1\omega \\ \frac{vm_1}{M\omega} =& A_1 \end{aligned} \qquad \begin{aligned} z_2(t) =& \frac{\ell}{2} + A_2 sin(\omega t) \\ \dot{z}_2(t) =& \omega A_2 cos(\omega t) \\ \dot{z}_2(t=0) = \frac{vm_2}{M} =& A_2\omega \\ \frac{vm_2}{M\omega} =& A_2 \end{aligned}$$

To find the position of m_1 at any time we simply add the center of mass position to the position of mass in the CM frame. For m_1 we obtain

$$\begin{aligned} z_{1T}(t) =& R(t) + z_1(t) \\ z_{1T}(t) =& \frac{\ell}{2} + \frac{m_1}{M}vt - \frac{1}{2}gt^2 + \frac{\ell}{2} + \frac{m_1 v}{M\omega}sin(\omega t) \\ z_{1T}(t) =& \ell + \frac{m_1}{M}vt - \frac{1}{2}gt^2 + \frac{m_2 v}{M\omega}sin(\omega t) \end{aligned} \tag{22}$$

And we do the same for the second mass. The resulting position function is then just

$$
\begin{aligned}
z_{2T}(t) =& R(t) + z_2(t) \\
z_{2T}(t) =& \frac{\ell}{2} + \frac{m_1}{M}vt - \frac{1}{2}gt^2 - \frac{\ell}{2} + \frac{m_2 v}{M\omega}sin(\omega t) \\
z_{2T}(t) =& \frac{m_1}{M}vt - \frac{1}{2}gt^2 + \frac{m_1 v}{M\omega}sin(\omega t)
\end{aligned} \tag{23}
$$

8 Problem 7.6

8.1 Part A

By using simple trigonometry of the vector representation of conservation of momentum of the lab and CM frame we can find the angle θ^*.

$$\alpha = \frac{1}{2}(\pi - \theta^*) \longrightarrow \theta^* = \pi - 2\alpha = \frac{\pi}{3} = 60^\circ$$

Using the same diagram we see the relation between masses and angles which can be manipulated to find m_2

$$
\begin{aligned}
tan\theta =& \frac{sin\theta^*}{\frac{m_1}{m_2} + cos\theta^*} \\
tan\theta\left(\frac{m_1}{m_2} + cos\theta^*\right) =& sin\theta^* \\
tan\theta\frac{m_1}{m_2} + tan\theta cos\theta^* =& sin\theta^* \\
tan\theta\frac{m_1}{m_2} =& sin\theta^* - tan\theta cos\theta^* \\
tan\theta m_1 =& m_2\left(sin\theta^* - tan\theta cos\theta^*\right) \\
\frac{tan\theta m_1}{(sin\theta^* - tan\theta cos\theta^*)} =& m_2 \\
\frac{m_1 tan56^\circ}{(sin60^\circ - tan56^\circ cos60^\circ)} =& m_2 = 11.88 \approx 12
\end{aligned} \tag{24}
$$

Where we have assumed m_1 is one unit of mass.

8.2 Part B

The fraction of kinetic energy imparted to the nucleus is given by

$$
\begin{aligned}
\frac{T_2}{T} =& \frac{4m_1m_2}{M^2}sin^2\frac{\theta^*}{2} \\
\frac{T_2}{T} =& \frac{4\cdot 1\cdot 12}{(12+1)^2}sin^2\frac{60^\circ}{2} \\
\frac{T_2}{T} =& \frac{48}{169}sin^2 30^\circ \\
\frac{T_2}{T} =& \frac{48}{169}\cdot\frac{1}{4} = 0.071
\end{aligned} \tag{25}
$$

9 Problem 7.7

9.1 Part A

Using the momentum vector triangle relation we see that the relation between the CM angle of scattering θ^* and the lab angle of recoil θ is

$$tan\theta = \frac{sin\theta^*}{\frac{m_1}{m_2} + cos\theta^*}$$

Taking the inverse tangent we obtain

$$\theta = tan^{-1}\left(\frac{sin70^\circ}{\frac{1}{7} + cos70^\circ}\right) = tan^{-1}(1.938) = 62.7^\circ \tag{26}$$

9.2 Part B

Using the relation between the CM angle of scattering and the labs angle of scattering we find

$$\alpha = \frac{1}{2}(180^\circ - \theta^*) = \frac{1}{2}(180^\circ - 70^\circ) = \frac{110^\circ}{2} = 55^\circ \tag{27}$$

9.3 Part C

The pions CM kinetic energy is $490KeV$. We know that the velocities of the proton and pion in the CM frame are given by

$$v_2' = \frac{m_1 u_1}{m_1 + m_2} \qquad v_1' = \frac{m_2 u_1}{m_1 + m_2}$$

We know that the initial kinetic in the CM frame is $T_0 = \frac{1}{2}m_1 u_1^2$. Now writing the final CM kinetic energy down we see that

$$T_1' = \frac{1}{2}m_1 v_1'^2 = \frac{1}{2}m_1\left(\frac{m_2 u_1}{m_1 + m_2}\right)^2 = \frac{m_1 u_1^2}{2}\left(\frac{m_2}{m_1 + m_2}\right)^2 = T_0\left(\frac{m_2}{m_1 + m_2}\right)^2$$

By knowing the initial kinetic energy in the CM frame and using this relation we find that the initial kinetic energy in the lab frame is

$$\begin{aligned}
T_1' &= T_0\left(\frac{m_2}{m_1 + m_2}\right)^2 \\
T_1'\left(\frac{m_1 + m_2}{m_2}\right)^2 &= T_0 \\
490KeV\left(\frac{1+7}{7}\right)^2 &= T_0 \\
490KeV\left(\frac{64}{49}\right) &= T_0 = 640KeV
\end{aligned} \tag{28}$$

10 Problem 7.8

10.1 Part A

In the center of mass frame after the decay the resultant particles are traveling in opposite directions with equal momenta.

$$P_1 = -P_2 \longrightarrow m_1 u_1 = -m_2 u_2 \longrightarrow u_1 = -\frac{m_2}{m_1} u_2$$

If the particle releases Q amount of energy the energy must be the two decay particles kinetic energies. The initial energy in the CM frame is $T = Q$ and after the decay the kinetic energy is $T_1 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$.

Substituting for one of the velocities and equating the initial energy to the final energy we find that

$$\begin{aligned}
T_i =& T_f \\
T_i =& T_1^* + T_2^* \\
Q =& \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \\
Q =& \frac{1}{2} m_1 \left(-\frac{m_2}{m_1} u_2 \right)^2 + \frac{1}{2} m_2 u_2^2 \\
Q =& \frac{1}{2} \frac{m_2^2}{m_1} u_2^2 + \frac{1}{2} m_2 u_2^2 \\
Q =& \frac{1}{2} m_2 u_2^2 \left(\frac{m_2}{m_1} + 1 \right) \\
Q =& \frac{1}{2} m_2 u_2^2 \left(\frac{m_2 + m_1}{m_1} \right) \\
Q =& T_2^* \left(\frac{M}{m_1} \right) \\
\frac{Q m_1}{M} =& T_2 *
\end{aligned} \tag{29}$$

Now by plugging this value into the conservation of energy equation we obtain

$$\begin{aligned}
T_i =& T_f \\
T_i =& T_1^* + T_2^* \\
Q =& T_1^* + Q \frac{m_1}{M} \\
Q - Q \frac{m_1}{M} =& T_1^* \\
Q \left(1 - \frac{m_1}{M} \right) =& T_1^* \\
Q \left(\frac{m_1 + m_2 - m_1}{M} \right) =& T_1^* \\
Q \frac{m_2}{M} =& T_1^*
\end{aligned} \tag{30}$$

10.2 Part B

The ratio of masses is $\frac{m_2}{m_1} = 4$ and the energy released is $Q = 1MeV$ and in the lab frame the unstable particle has $2.25MeV$. The maximum amount of kinetic energy that m_1 could have is when m_2 has none, which means the particle 2 has no kinetic energy. Henceforth, we see that the max kinetic energy will just be the sum

$$T_0 = 3.25MeV = T_1 + T_2 \longrightarrow T_1 = 3.25MeV \tag{31}$$

And when particle one has the least energy is when it has $T = \frac{m_1}{M}Q$ and particle has the rest of the energy. This is to account for conservation of momentum in both reference frames. Hence we find the minimum energy for particle one is

$$T_{min} = \frac{m_1}{M}Q = \frac{4}{5}1MeV = 0.8MeV \tag{32}$$

11 Extra Credit Problems

11.1 Problem 4.23

11.1.1 Part A

Using the results from problem 4.21 and 4.22 we see that the velocity of the spaceship if the slingshot maneuver was to put the spaceship in a direction normal to the orbital plane of jupiter will be the same as if it were to be redirected radially outward as in problem 4.22. We see that this is true because it is still making $90°$ redirection angle to its original path. The only difference is that the velocity vectors exist in the $x - z$ plane instead of the $x - y$ plane. The resulting velocity is then

$$\vec{v} = \left\langle 13.1\hat{\imath} + 0\hat{\jmath} + 5.7\hat{k} \right\rangle \tfrac{km}{s} \longrightarrow v = \sqrt{5.7^2 + 13.1^2}\tfrac{km}{s} = 14.3\tfrac{km}{s} \qquad \theta = tan^{-1}\left(\frac{5.7\frac{km}{s}}{13.1\frac{km}{s}}\right) = 23.5° \tag{33}$$

11.1.2 Part B

We know that if the spaceship is slingshotted perpendicular to the orbital plane it will have no radial velocity and hence be at the perihelion. The velocity will be entirely tangential. Therefore, the angular momentum is the magnitude of the velocity times the jupiter sun distance times the satellites mass. Just as in problem 4.22 we use the radial equation we obtain the initial energy. Also $\dot{r}$ is zero at aphelion.

$$\begin{aligned}
E_i =& \frac{m^2 v_i^2 R_J^2}{2mR_J^2} - G\frac{Mm}{R_J} \\
E_i =& m\left(\frac{v_J^2}{2} - G\frac{M}{R_J}\right) \\
E_i =& \frac{m}{2}\left((14.3\tfrac{km}{s})^2 - 2G\frac{1.99 \times 10^{30}kg}{5.2 \cdot 1.5 \times 10^{11}m}\right) \\
E_i =& -6.792 \times 10^7 J
\end{aligned}$$

$$\begin{aligned}
E_f =& \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mR_a^2} + V(R_a) \\
E_f =& \frac{m^2 v_J^2 R_J^2}{2mR_a^2} - G\frac{Mm}{R_a} \\
E_f =& \frac{mv_J^2 R_J^2}{2R_a^2} - G\frac{Mm}{R_a}
\end{aligned}$$

Now we can equate initial energy to final energy to obtain.

$$
\begin{aligned}
E_i =& E_f \\
E_i =& \frac{mv_J^2 R_J^2}{2R_a^2} - G\frac{Mm}{R_a} \\
E_i R_a^2 =& \frac{mv_J^2 R_J^2}{2} - GMmR_a \\
2E_i R_a^2 + 2GMR_a - v_J^2 R_J^2 =& 0
\end{aligned}
$$

Where there is a common factor, m, that is divided out on both sides. By solving this quadratic with the obtain

$$
R_a = \frac{-GM \pm \sqrt{G^2M^2 - 4(2E_i)(-v_J^2 R_J^2)}}{2(2E_i)} = 1.15 \times 10^{12} m \approx 7.8 AU \tag{34}
$$