Math 23A Practice Midterm 2

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Problem 1

Problem 1 Part A

Let $\vec{r}(t) = (x(t), y(t))$ be a parametrization for the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ with $\vec{r}(0) = (2, 0)$. Let $f(x, y) = x^3 + xy + y^3$ and $\vec{R}(t) = (x(t), y(t), f \circ \vec{r}(t))$. Find the equation for the tangent line to $\vec{R}(t)$ at $t = 0$.

Solution to Problem 1 Part A

The paramtrization is $\vec{r}(\theta) = (x(t), y(t)) = (2\cos\theta, 3\sin\theta)$ and $\vec{r}'(\theta) = (-2\sin(\theta), 3\cos\theta) \longrightarrow \vec{r}'(0) = (0, 3)$. The tangent line is going to be given by $T(t) = \vec{R}(0) + \vec{R}'(0)t$, so then we find

$$\vec{R}(t = 0) = (x(0), y(0), f(\vec{r}(0)) = (2, 0, f(2, 0)) = (2, 0, 2^3 + 0 + 0) = (2, 0, 8)$$

Next we find

$$\vec{R}'(t = 0) = (x'(0), y'(0), Df(\vec{r}(0)) = (0, 3, \nabla f(\vec{r}(0)) \cdot \vec{r}'(0)) = (0, 3, \nabla f|_{(2,0)} \cdot (0, 3)) = (0, 3, 6)$$

where $\nabla f = (3x^2 + y, x + 3y^2) \longrightarrow \nabla f|_{(2,0)} = (3(4) + 0, 2 + 0) = (12, 2)$ and $\nabla f|_{(2,0)} \cdot (0, 3) = 6$

Therefore,

$\boxed{\text{T(t)=(2,0,8)+t(0,3,6)}}$

Problem 1 Part B

Let $f(x, y) = (2 + xy^2, x^2 + y^2, 3 + y^3)$ and $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Compute $\mathrm{D}(f \circ T)(2, \frac{\pi}{2})$.

Solution to Problem 1 Part B

$$\begin{aligned}
D(f \circ T)(2, \tfrac{\pi}{2} &= \nabla f(T(2, \pi/2)) \cdot T'(2, \pi/2) \\
&= \nabla f(0, 2) \cdot T'(2, \pi/2) \\
&= \begin{pmatrix} y^2 & 2xy \\ 2x & 2y \\ 0 & 3y^2 \end{pmatrix}_{0,2} \cdot \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}_{2,\pi/2} \\
&= \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 12 \end{pmatrix} \cdot \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \\
&= \boxed{\begin{pmatrix} 0 & -8 \\ 4 & 0 \\ 12 & 0 \end{pmatrix}}
\end{aligned}$$

Problem 2

You are walking on a surface given by $Q(x,y) = y\sin(\pi x) - x\sin(\pi y) + 10$

Problem 2 Part A

From the point (2,1,10), what x, y-direction should you follow to go up the fastest ?

Solution to Problem 2 Part A

The direction of fastest increase is the gradient. Hence,

$$\nabla Q(2,1) = (y\pi\cos(\pi x) - \sin(\pi y), \sin(\pi x) - \pi x\cos(\pi y))_{2,1} = \boxed{(\pi, 2\pi)}$$

Problem 2 Part B

Find an x, y-direction from (2,1,10) that will keep you at the same level.

Solution to Problem 2 Part B

We want the directional derivative to be zero therefore,

$$\begin{aligned} \nabla f \cdot (a,b) =& 0 \\ (\pi, 2\pi) \cdot (a,b) =& 0 \\ a\pi + 2b\pi =& 0 \\ a =& -2b \end{aligned}$$

Any (a,b) that satisfies this last relationship will work the simplest is $\boxed{(a,b) = (-2,1)}$

Problem 2 Part B

What slope will you encounter as you set off from (2,1,10) towards (-1,7,10)?

Solution to Problem 2 Part C

We first need to find our direction vector **and make it unit length**.

$$\vec{d} = (-1-2, 7-1) = (-3,6) \longrightarrow \hat{d} = \frac{\vec{d}}{||\vec{d}||} = \tfrac{1}{\sqrt{45}}(-3,6)$$

Now all we do is dot this vector into the gradient to get the slope in the direction we want (definition of a directional derivative!)

$$\nabla f(2,1) \cdot \hat{d} = \frac{(\pi, 2\pi) \cdot (-3,6)}{\sqrt{45}} = \boxed{\frac{9\pi}{\sqrt{45}}}$$

Problem 3

Problem 3 Part A

What is the rate of change of $f(x, y, z) = e^x \cos z - y$ at (1,1,1) in a direction normal to $x^2 z + z^y = 2$ at (1,1,1)?

Solution to Problem 3 Part A

First let us find the normal to the graph ,

$$\vec{n} = \nabla(x^2 z + z^2 y) = (2xz, z^2, x^2 + 2zy)_{1,1,1} = (2, 1, 3) \longrightarrow \hat{n} = \frac{\vec{n}}{||\vec{n}||} = \frac{1}{\sqrt{14}}(2, 1, 3)$$

Then we find the directional derivative

$$\begin{aligned} D_{\hat{n}} f(1,1,1) =& \nabla f(1,1,1) \cdot \hat{n} \\ =& (e^x \cos z - 1, -1, e^x \sin z)_{1,1,1} \cdot \hat{n} \\ =& (e \cos(1), -1, -e \sin(1)) \cdot \frac{1}{\sqrt{14}}(2, 1, 3) \\ =& \boxed{\frac{2e[\cos(1) - \sin(1)] - 1}{\sqrt{14}}} \end{aligned}$$

Problem 3 Part B

Find the second order Taylor polynomial for $f(x, y) = y^2 e^{-x^2}$ The second order Taylor polynomial approximation is given by

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial^2 f}{\partial x^2}\frac{(x - x_0)^2}{2!} + \frac{\partial^2 f}{\partial y^2}\frac{(y - y_0)^2}{2!} + \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0)$$

So all we need to do is calculate the individual terms with $(x_0, y_0) = (1, 1)$

$$\begin{aligned} f(1,1) =& \frac{1}{e} \\ \frac{\partial f}{\partial x} =& -2xy^2 e^{-x^2} = -\frac{2}{e} \\ \left.\frac{\partial f}{\partial y}\right|_{1,1} =& \left.2ye^{-x^2}\right|_{1,1} = \frac{2}{e} \\ \left.\frac{\partial^2 f}{\partial x^2}\right|_{1,1} =& \left.-2y^2 e^{-x^2} + 4x^2 y^2 e^{-x^2}\right|_{1,1} = \frac{2}{e} \\ \left.\frac{\partial^2 f}{\partial y^2}\right|_{1,1} =& \left.2e^{-x^2}\right|_{1,1} = \frac{2}{e} \\ \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{1,1} =& \left.-4xye^{-x^2}\right|_{1,1} = -\frac{4}{e} \end{aligned}$$

Putting these all together we get

$$\boxed{f(x, y) \approx \frac{1}{e} - \frac{2}{e}(x - 1) + \frac{2}{e}(y - 1) + \frac{1}{e}(x - 1)^2 + \frac{1}{e}(y - 1)^2 - \frac{4}{e}(x - 1)(y - 1)}$$

Problem 4

Find and classify all critical points for the function $G(x,y) = \frac{1}{4}x^4 - \frac{5}{3}x^3 + y^3 + 3x^2 - \frac{3}{2}y^2 + 20$

Solution to Problem 5

There are 3 y critical points and 2 x critical points, they are y=0,1 and x=0,3,2. It is then your job to find the second derivative matrix and look at the concavity (i.e. is the point in question a minimum, maximum or saddle point?). I am out of time to do it now but I will try to do it later. There was 2 major problems on my second midterm about classifying critical points. I have not seen a lot of these problems for your class so maybe this is just a waste of time... but this material is important regardless.