Method for Approximating Irrational Numbers

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Abstract

I will put forth an algorithm for producing increasingly accurate rational approximations for the square root of any natural number that is not a perfect square.

1 Theorem

If n is a natural number and $\sqrt{n}$ is rational, then $\sqrt{n}$ is in fact a natural number itself. We say in this case that n is a *perfect square*. The *contrapositive* of this statement is the fact that if the natural number n is not a perfect square, then $\sqrt{n}$ is not a rational number.[1]

The simple iterative algorithm described below generates increasingly accurate rational approximations to the (irrational) square root of a given non-perfect square. We assume that n is not a perfect square.[2] We also choose a desired degree of accuracy, i.e., we choose some $\varepsilon > 0$. The boxed algorithm below produces a rational number r^* satisfying

$$|r^* - \sqrt{n}| < \varepsilon.$$

- Let r_0 be the (unique) natural number satisfying $(r_0 - 1)^2 < n < r_0^2$.
- Iterative step: Assuming that $k \geq 1$ and r_{k-1} has been calculated, let

$$r_k = \frac{1}{2}\left(r_{k-1} + \frac{n}{r_{k-1}}\right).$$

- If $|r_k^2 - n| \leq \varepsilon$, set $r^* = r_k$. If $|r_k^2 - n| > \varepsilon$, repeat the iterative step.

[1] This is another generalization of the fact that $\sqrt{2}$ is irrational.
[2] This implies in particular that $n > 1$.

2 Proof

Proposition 1. *If n is a natural number but not a perfect square, then $\sqrt{n}$ is not rational.*

Proof. Let k be a natural number that is not a perfect square such that $\sqrt{k} = \frac{p}{q}$, where p and q are integers and $gcd(p,q) = 1$. Squaring both sides we get $k = \frac{p^2}{q^2} = \frac{p}{q} \cdot \frac{p}{q}$. This means that $q^2 \mid p$ or $q \mid p^2$. Since p is not a multiple of q there is no integer j such that $p = j \cdot q$ hence, $k \neq \frac{(j \cdot q)(j \cdot q)}{q \cdot q}$ for any $j \in \mathbb{Z}$. The only way k can be natural number is when $q^2 = q = 1$ hence $k = p^2$. Therefore k is a perfect square, which contradicts our assumption. □

This conclusion motivates the need to approximate the square root of non-perfect squares. One such way is the algorithm I have proposed above.

2.1 The Algorithm

Here we will layout a series of steps to obtain an approximation of the irrational root to as much as accuracy as desired.

1. Choose a positive ε for the coveted precision.
2. Pick a natural number, r_0, that satisfies $(r_0 - 1)^2 < n < r_0^2$.
3. Obtain r_1 from $r_k = \frac{1}{2}\left(r_{k-1} + \frac{n}{r_{k-1}}\right)$ where $r_{k-1} = r_0$.
4. Check to see if the termination condition, $|r_1^2 - n| < \varepsilon$, holds true. If it is not true continue to step 5, otherwise we have obtained an appropriate approximation that we define as r^* .
5. Obtain r_k from $r_k = \frac{1}{2}\left(r_{k-1} + \frac{n}{r_{k-1}}\right)$ where r_{k-1} is the value previously did not meet the termination condition. For each new value of r_k check to see if $|r_k^2 - n| < \varepsilon$ is true. Repeat this step if the termination condition is not met

2.1.1 Existence and Uniqueness

Lemma 2. *For all $n \in \mathbb{N}$ where n is not a perfect square, there exists a unique natural number, r_0, satisfying $(r_0 - 1)^2 < n < r_0^2$.*

Proof. Let $A_n = \{r \in \mathbb{N} \mid r^2 > n\}$, where $A_n \subseteq \mathbb{N}$ and n is not a perfect square. Since $n > 1$, $n^2 > n$ so $n \in A_n$. Hence A_n is nonempty. By the Well Ordering Principle A_n has a least element, which we will define as r_0. Note that $(r_0 - 1) < r_0$ and that r_0 is the least element of A_n. Therefore $(r_0 - 1)$ is not in A_n. Since n is not a perfect square, $(r_0 - 1)^2 \neq n$. If $(r_0 - 1) \geq n$ then it is the first perfect square larger than n so $(r_0 - 1) = r_0$, which is not true. Thus $(r_0 - 1)^2 < n < r_0^2$. □

2.1.2 Rational Termination

Proposition 3. *All successive approximations are rational numbers.*

Proof. We will prove this proposition by induction. Our base case is $k = 1$ so

$$r_1 = \frac{1}{2}\left(r_0 + \frac{n}{r_0}\right) = \frac{1}{2}\left(\frac{r_0^2}{r_0} + \frac{n}{r_0}\right) = \left(\frac{r_0^2 + n}{2r_0}\right)$$

which is a rational number because $r_0^2 + n$ and $2r_0$ are integers. Now we must show that r_k being rational implies r_{k+1} is rational.

$$r_k = \frac{1}{2}\left(r_{k-1} + \frac{n}{r_{k-1}}\right) = \left(\frac{r_{k-1}^2 + n}{2r_{k-1}}\right)$$

which is a rational number.

$$r_{k+1} = \frac{1}{2}\left(r_k + \frac{n}{r_k}\right) = \left(\frac{r_k^2 + n}{2r_k}\right)$$

Using our definition of r_k we obtain

$$r_{k+1} = \left(\frac{\left(\frac{r_{k-1}^2 + n}{2r_{k-1}}\right)^2 + n}{2\left(\frac{r_{k-1}^2 + n}{2r_{k-1}}\right)}\right) = \left(\frac{\left(r_{k-1}^2 + n\right)^2 + 2nr_{k-1}}{2(r_{k-1})(r_{k-1}^2 + n)}\right)$$

By induction all successive approximations are rational. □

2.1.3 Termination in a Finite Number of Iterations

Lemma 4. *For $k \geq 1$,*

$$r_k^2 - n = \left(\frac{r_{k-1}^2 - n}{2r_{k-1}}\right)^2$$

Proof.

$$r_k = \frac{1}{2}\left(r_{k-1} + \frac{n}{r_{k-1}}\right) = \left(\frac{r_{k-1}^2}{2r_{k-1}} + \frac{n}{2r_{k-1}}\right) = \left(\frac{r_{k-1}^2 + n}{2r_{k-1}}\right)$$

Square both sides

$$r_k^2 = \left(\frac{r_{k-1}^2 + n}{2r_{k-1}}\right)^2$$

Subtract n from both sides

$$r_k^2 - n = \left(\frac{\left(r_{k-1}^2 + n\right)^2}{4r_{k-1}^2}\right) - n = \left(\frac{\left(r_{k-1}^4 + 2nr_{k-1}^2 + n^2\right)}{4r_{k-1}^2}\right) - \left(\frac{4nr_{k-1}^2}{4r_{k-1}^2}\right) =$$

Simplifying and completing the square we obtain

$$r_k^2 - n = \left(\frac{\left(r_{k-1}^4 + 2nr_{k-1}^2 + n^2 - 4nr_{k-1}^2\right)}{4r_{k-1}^2} \right) = \left(\frac{\left(r_{k-1}^4 - 2nr_{k-1}^2 + n^2\right)}{4r_{k-1}^2} \right)$$
$$= \left(\frac{\left(r_{k-1}^4 - 2nr_{k-1}^2 + n^2\right)}{4r_{k-1}^2} \right)$$
$$= \left(\frac{\left(r_{k-1}^2 - n\right)^2}{4r_{k-1}^2} \right) = \left(\frac{r_{k-1}^2 - n}{2r_{k-1}} \right)^2$$

□

Conjecture 5. *For $k \geq 1$*

$$|r_k^2 - n| \leq |r_0^2 - n| \left(\tfrac{1}{4}\right)^k$$

Proof. We will prove this by induction. Observe that $|r_k^2 - n| \leq r_k^2$ so $\frac{|r_k^2 - n|}{r_k^2} \leq 1$. The base case is $k = 1$

$$|r_1^2 - n| = \left(\frac{r_0^2 - n}{2r_0} \right)^2 = \frac{\left(r_0^2 - n\right)^2}{r_0^2} \left(\frac{1}{4} \right)^1 = |r_0^2 - n| \left(\tfrac{1}{4}\right)^1 \left(\tfrac{|r_0^2 - n|}{r_0^2} \right) \leq |r_0^2 - n| \left(\tfrac{1}{4}\right)^1 \cdot 1$$

Using our observation along with lemma 4 we see that

$$|r_{k+1} - n| = \frac{\left(r_k^2 - n\right)^2}{4r_k^2} = \left(\frac{|r_k^2 - n|}{r_k^2} \right) \left(|(r_k^2 - n| \cdot \frac{1}{4} \right) \leq \left(|r_k^2 - n| \cdot \frac{1}{4} \right)$$

Using our inductive hypothesis we obtain

$$|r_{k+1}^2 - n| \leq |r_0^2 - n| \left(\tfrac{1}{4}\right)^k \left(\tfrac{1}{4}\right) = |r_0^2 - n| \left(\tfrac{1}{4}\right)^{k+1}$$

□

Proposition 6. *The termination condition $|r_k^2 - n| \leq \varepsilon$ will always be met after a finite number of steps.*

Proof. For any $k \in \mathbb{N}$ we see that $0 \leq |r_k^2 - n|$ and from conjecture 5 we see that $|r_k^2 - n| \leq |r_0^2 - n| \left(\frac{1}{4}\right)^k$. Therefore

$$0 \leq |r_k^2 - n| \leq |r_0^2 - n| \left(\tfrac{1}{4}\right)^k$$

We see that $\lim_{k\to\infty} |r_0^2 - n| \left(\frac{1}{4}\right)^k = 0$ since $|r_0^2 - n|$ is a constant and $\left(\frac{1}{4}\right)^k$ is a converging geometric series. By the Squeeze Theorem $\lim_{k\to\infty} |r_k^2 - n|$ must converge to zero at least as quickly as $\lim_{k\to\infty} |r_0^2 - n| \left(\frac{1}{4}\right)^k$ converges to zero. This means that only after a infinite amount iterations will our approximation become exact. Consequently, there will be a finite amount of iterations to obtain an approximation of any finite accuracy. □

2.2 Upper Bound to Number of Iterations

Proposition 7. *There is a finite upper bound on the number of iterations required to obtain the desired accuracy of the approximation.*

Proof. Combining the condition for termination and the result of conjecture 5 we obtain

$$|r_0^2 - n| \left(\tfrac{1}{4}\right)^k \leq \varepsilon$$

By solving for k we find the number of iterations needed for a desired accuracy.

$$\left(\tfrac{1}{4}\right)^k \leq \frac{\varepsilon}{|r_0^2 - n|}$$

Take the logarithm base $\frac{1}{4}$ of both sides

$$k \leq \log_{\frac{1}{4}} \left| \frac{\varepsilon}{r_0^2 - n} \right|$$

Consequently, the number of iterations, k, is always finite under the algorithm conditions, $\varepsilon \neq 0$ and $r_0^2 \neq n$. □

3 Examples

Let us approximate the square root of 17 to a degree of accuracy of $\frac{1}{10}$ using our algorithm (step 1). First we see that the closest perfect square larger than 17 is 25. This will be our r_0^2 hence, $r_0 = 5$ which satisfies the condition of step 2. We also see that the next lowest perfect square is 16, from here we can conclude that $4 < \sqrt{17} < 5$. We begin by plugging into our equation in step 3.

$$r_1 = \frac{1}{2}\left(5 + \frac{17}{5}\right) = \frac{1}{2}\left(\frac{25}{5} + \frac{17}{5}\right) = \frac{1}{2}\left(\frac{25+17}{5}\right) = \frac{1}{2}\left(\frac{42}{5}\right) = \frac{21}{5}$$

We now must check if our r_1 is an accurate enough approximation as required by step 4.

$$|\left(\tfrac{21}{5}\right)^2 - 17| < \tfrac{1}{10}$$

$$|\tfrac{441}{25}) - \tfrac{(17)(25)}{25}| = |\tfrac{441-425}{25}| < \tfrac{1}{10}$$

$$|\tfrac{16}{25}| \not< \tfrac{1}{10}$$

We must plug our new value, into the equation of step 3.

$$r_2 = \frac{1}{2}\left(\frac{21}{5} + \frac{17}{\frac{21}{5}}\right) = \frac{1}{2}\left(\frac{(21)(21)}{(5)(21)} + \frac{(17)(5)(5)}{(21)(5)}\right)$$

$$r_2 = \frac{1}{2}\left(\frac{441}{105} + \frac{425}{105}\right) = \frac{1}{2}\left(\frac{441+425}{105}\right) = \frac{433}{105}$$

We now must check if our r_1 is an accurate enough approximation as required by step 4.

$$|\left(\frac{433}{105}\right)^2 - 17| < \frac{1}{10}$$

$$|\frac{187489}{11025}) - \frac{(17)(11025)}{11025}| = |\frac{187489-187425}{11025}| = |\frac{64}{11025}| < \frac{1}{10}$$

Let us obtain common denominators to clearly see that this is a true statement.

$$|\frac{(64)(10)}{(11025)(10)}| < \frac{(1)(11025)}{(10)(11025)}$$

$$|\frac{640}{110250}| < \frac{11025}{110250}$$

It is obvious to see that our approximation is well within our desired accuracy so we can define $r^* = r_2 = \frac{433}{105}$.