Solution to Question 13 of Homework 1

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1 Question

A physical pendulum consists of an 85cmlong, 240 g mass, uniform wooden rod hung from a nail near one end. The motion is damped because of friction in the pivot; the damping force is approximately proportional to . The rod is set in oscillation by displacing it 15 from its equilibrium position and releasing it. After 8.1° , the amplitude of the oscillation has been reduced to 5.8°.

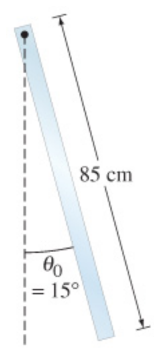

Figure 1: The physical pendulum of the problem. Courtesy of Giancoli.

2 Solving the EOM

Solving this differential equation is a little advanced for this course (will learn to do this in Physics 116B) so therefore I present to you the solution and you can check for yourself that it is indeed a solution. The solution is

$$\theta(t) = Ae^{-\gamma t}cos(\omega' t) \qquad (14-16)$$

Now we must solve for the constants. A is obviously amplitude in degrees and will be the maximum amplitude of the motion, hence it is the starting amplitude, $A = 15°$.

From the book we can determine what the other constants are. These constants come as a consequence of solving the second order linear differential EOM.

3 Part A: Determine γ

Since we know what the amplitude ($A = 5.8°$) is at a certain time ($t = 8.1s$) we can find what γ is. We can think of $Ae^{-\gamma t}$ as maximum possible amplitude (as a function of time) of the SHO. Since the question gives us maximum amplitude at a certain time we can determine γ by algebraic manipulation of the amplitude function.

$$\begin{aligned}
\theta(5.8s) =& 15° e^{-\gamma 5.8s} \\
8.5° =& 15° e^{-\gamma 5.8s} \\
\frac{8.5°}{15°} =& e^{-\gamma 5.8s} \\
ln\left(\frac{8.5°}{15°}\right) =& -\gamma 5.8s \\
-\frac{1}{5.8s} ln\left(\frac{8.5°}{15°}\right) =& \gamma = 0.12s^{-1}
\end{aligned}$$

4 Part B: Determine Period, T

Now we must determine the frequency, and hence period, of the oscillator. Equation (14-18) gives us the equation of the frequency for a SHO of a spring and mass. However, the only difference for us is that $\frac{k}{m}$ is now $\frac{mgd}{I}$

Where does this come from? It comes from knowing what the moment of inertia is for a rod about one end and the fact that the torque acts on the rod at its midpoint (or center of mass). Hence,

$$\omega = \sqrt{\frac{mg\frac{\ell}{2}}{\frac{1}{3}m\ell^2}} = \sqrt{\frac{3g}{2\ell}}$$

$$\begin{aligned}
\omega' =& \sqrt{\frac{mg\frac{3g}{2\ell}}{-}\gamma^2} \\
\omega' =& \sqrt{\frac{3 \cdot 9.8}{2 \cdot 0.85} - 0.12^2} \\
\omega' =& 4.16s^{-1}
\end{aligned}$$

Now we just invert and multiply by 2π to obtain the period.

$$T = \frac{2\pi}{\omega'} = 1.51s$$

5 Part C: Determine time when amplitude is halved

Same story here. We use manipulation just as in part A to determine (except we are isolating t instead of γ) the time where maximum amplitude function is $\frac{A}{2}$.

$$\begin{aligned} \theta(t) = \frac{A}{2} &= Ae^{-\gamma t} \\ \frac{1}{2} &= e^{-\gamma t} \\ ln\left(\frac{1}{2}\right) &= -\gamma t \\ \frac{1}{\gamma} ln(2) &= t = 5.8 sec \end{aligned}$$

6 Graph of Damped Harmonic Oscillator

It is very nice to look at your solution as plot for these problems. Here I have plotted the angular displacement and the maximum amplitude function. You can see that the maximum amplitude touches the angular displacement graph once every period at it's max amplitude point. You can also verify your results with the plots results to make sure your solution is correct.

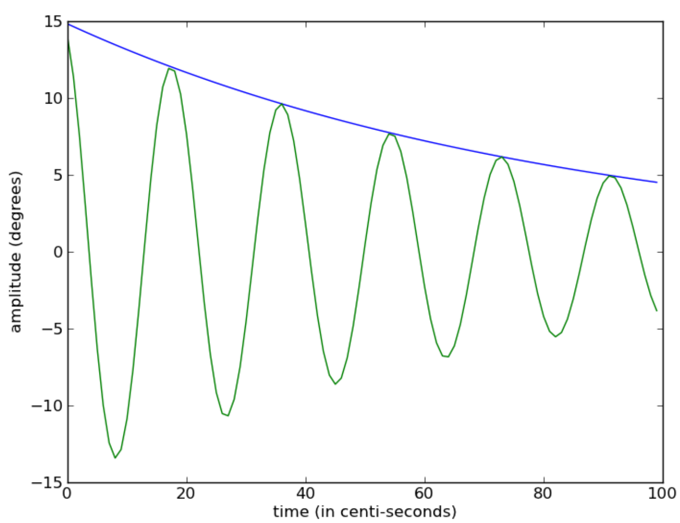

Figure 2: The physical pendulum angular displacement as a function of time. Plot made with SciPy.