

Physics 6C Review 3

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1 Magnetism

1.1 Biot-Savart Law

The Biot-Savart law is useful for determining the magnetic field due to a known arrangements of current. Since magnetic fields add vectorially we can find the magnetic field due to multiple current carrying wires one by one (pretending all other wires are not there). Then just add all vectors together and the result is the overall magnetic field. For Physics 6C the Biot-Savart law is unnecessary and difficult and the results of certain situations should simply be memorized. The Biot-Savart Law is as stated

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{r^2} \quad (1)$$

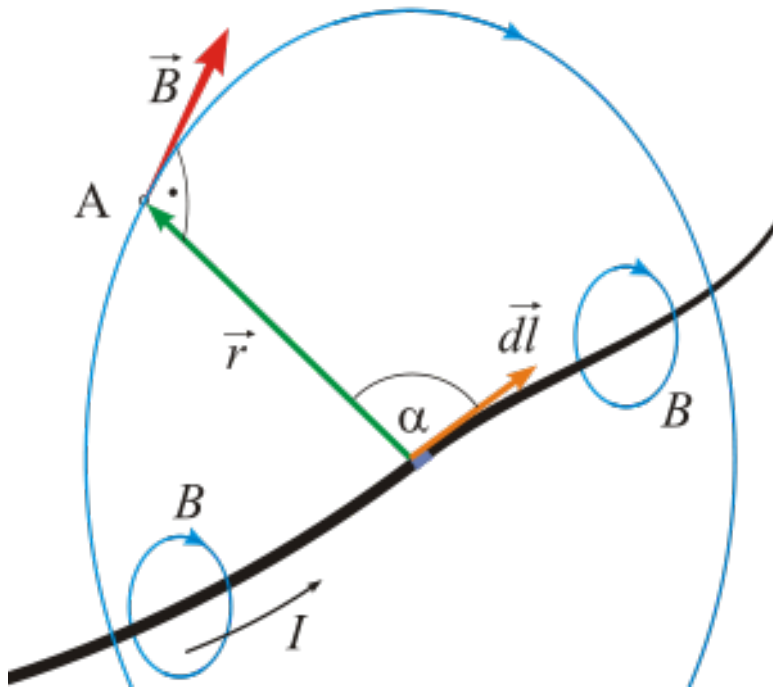


Figure 1: Biot-Savart Law is used to determine the magnetic field at any point in space due exclusively to current.

The results below are stated without derivation and are the most important.

1.1.1 Magnetic Field Due to Straight Current Carrying Wire.

The magnetic field, \vec{B} , a distance, r , away caused by straight wire carrying current, I , is given by equation 2.

$$B = \frac{\mu_0 I}{2\pi r} \quad (2)$$

1.1.2 Magnetic Field Due to A Loop of Current Carrying Wire

Along the center axis center of the loop the magnetic field is determined by

$$B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{\frac{3}{2}}} \quad (3)$$

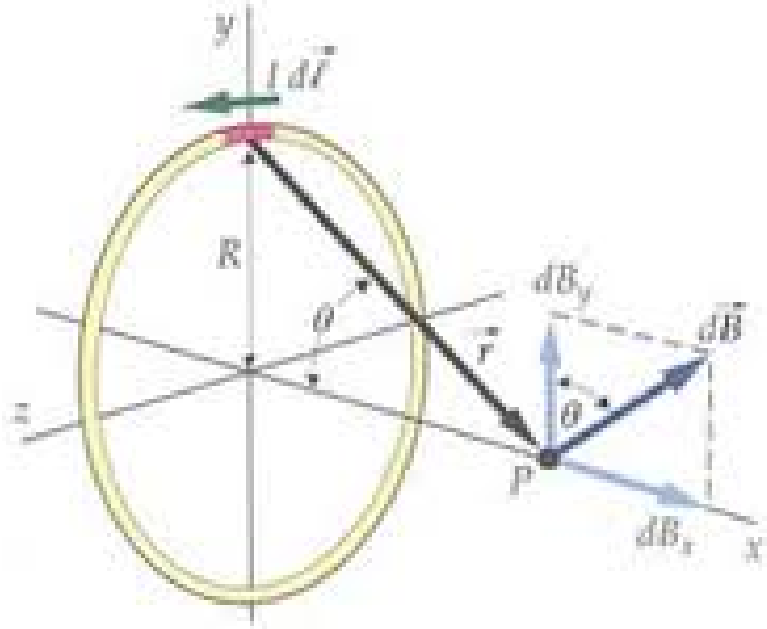


Figure 2: Calculating the Magnetic field inside one coil of current carrying wire using Biot-Savart law.

Where R is the radius of the loop(s) and x is the distance along the center axis from the loops center. The previous equation is for ANY point along the center axis of the loop. For the direct center of the loop(s) we just look at the limiting case as $x \rightarrow 0$.

$$\begin{aligned} B &= \frac{\mu_0 I R^2}{2(R^2 + x^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0 I R^2}{2(R^2 + 0^2)^{\frac{3}{2}}} \\ &= \frac{\mu_0 I R^2}{2R^2 \cdot \frac{3}{2}} \\ &= \frac{\mu_0 I R^2}{2R^3} \\ B &= \frac{\mu_0 I}{2R} \end{aligned} \quad (4)$$

Now if we look at the case as $x \rightarrow \infty$, or $x \gg R$, we get

$$B \approx \frac{\mu_0 I}{2x^3} \rightarrow 0 \quad (5)$$

Since as x gets very large, it becomes more dominant in the equation and since its in the denominator it makes the magnetic field approach zero.

2 Amperes Law

Amperes Law is very similar to Gauss' Law except that Amperes relates line integral to a surface integral and Gauss' relates a surface integral to a volume integral. Amperes states that if you integrate a **closed** path in an magnetic vector field the result is just the total current enclosed by the closed loop and scaled by a factor $\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$. Below we see the difference between Gauss Law

$$\oint_A \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

and Amperes Law

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_{Through} \quad (6)$$

Where A is the area being integrated in Gauss' Law and C is the path being integrated in Amperes Law. ***This law is true for all cases, but is only useful in cases of high symmetry.*** There are a few cases where it is useful one of which is stated below.

2.1 Straight Wire

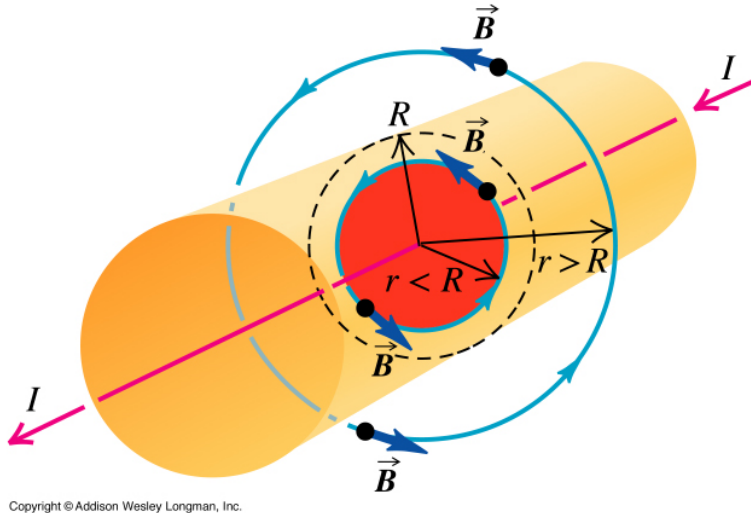


Figure 3: Amperes Law diagram. Magnetic field around current-carrying wire in perfect circles.

2.1.1 Outside of the Wire

We know that a current carrying wire produces a circular magnetic field around the wire. If we choose an integration path that is along an arbitrary magnetic field (which is in a circle) at a constant radius we get $\oint_C \vec{B} \cdot d\vec{\ell} = \vec{B}2\pi r$. By Amperes Law this would give us

$$B = \frac{\mu_0 I}{2\pi r} \quad (7)$$

Which is exactly equation 2. If we already knew the magnetic field at a given radius we could determine the current in the wire to be $I = \frac{B \cdot 2\pi r}{\mu_0}$

2.1.2 Inside of Wire

If we are given the radius, R , of the wire transporting the current, I , then we can find the magnetic field inside the wire at any distance, r , from the center of the wire. To find the total current passing through our loop we use the definition of current density, $\vec{j} = \frac{I}{A_{total}}$, then use \vec{j} to find the current enclosed by

$$I_{through} = \vec{j} \cdot \vec{A} = I \frac{A_{enclosed}}{A_{total}}$$

Then just apply Amperes Law to obtain

$$\begin{aligned} \oint_C \vec{B} \cdot d\vec{\ell} &= \mu_0 I_{through} \\ B 2\pi r &= \mu_0 I_{through} \\ B &= \frac{I \mu_0 \pi r^2}{(2\pi r)(\pi R^2)} \\ B &= \frac{I \mu_0 r}{2\pi R^2} \end{aligned} \tag{8}$$

3 Electromagnetic Induction and Faraday's Law

After discovering that a current creates magnetic field, scientists began to wonder if a magnetic field could create a current. Around 1820 Joseph Henry and Michael Faraday independently had discovered that a changing magnetic field would induce a current but not a constant magnetic field. Faraday did many experiments to verify his results that a changing magnetic field induced an electromotive force and alternately changing the area of the coil induces an electromotive force. The thing that matters is the relative motion of the coil and magnet.

3.1 Magnetic Flux and Faraday's Law of Induction

The magnetic flux, just like electric flux, is defined as the scalar product (dot product) of the magnetic field and the area vector.

$$\phi_B = \vec{B} \cdot \vec{A}$$

From Faraday's experiments and subsequent derivations he has found that for N number of coils the induced electromotive force, \mathcal{E} , is defined as

$$\mathcal{E} = -\frac{d\phi_b}{dt} \tag{9}$$

This states that a current produced by an induced electromotive force moves in a direction so that the magnetic field created by that current opposes the original change in the flux. This is known as Lenz's Law

3.1.1 Lenz's Law

There are now TWO different magnetic fields.

1. The changing magnetic field (or flux) that induces the current
2. The magnetic field produced by the induced current. This field opposes change in the first.

We can also restate Lenz's Law: **An induced electromotive force is always in a direction that opposes the original change in magnetic flux that caused it.** Lenz's Law is used to determine the direction of the conventional current induced in a loop due to a change in magnetic flux inside the loop. To produce an induced current you need a closed conducting loop and a changing external magnetic flux. If these conditions are met then the steps to use Lenz's Law are:

1. determine the change in magnetic flux ($\phi_B = BA\cos(\theta)$) inside the loop is decreasing, increasing, or unchanged.
2. The magnetic field due to the induced current: (a) points in the same direction as the external field if the flux is decreasing; (b) points in the opposite direction from the external field if the flux is increasing (c) is zero if the flux is not changing.
3. Once you know the direction of the induced magnetic field, use right-hand-rule to find the direction of the induced current.
4. Always keep in mind that there are two magnetic fields: (1) an external field whose flux must be changing if it is to induce an electric current, and (b) a magnetic field produced by the induced current.

3.1.2 A changing Magnetic Flux Produces an Electric Field

We know that voltage (or equivalently electromotive force) is the line integral of the electric field, the induced electric field. We state Faraday's Law in a general form as

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_B}{dt} \quad (10)$$

3.2 Electric Motor/Generator

A simple DC motor has a coil of wire that can rotate in a magnetic field. The current in the coil is supplied through two brushes that make moving contact with a split ring. The coil lies in a steady magnetic field. The forces exerted on the current-carrying wires create a torque on the coil.

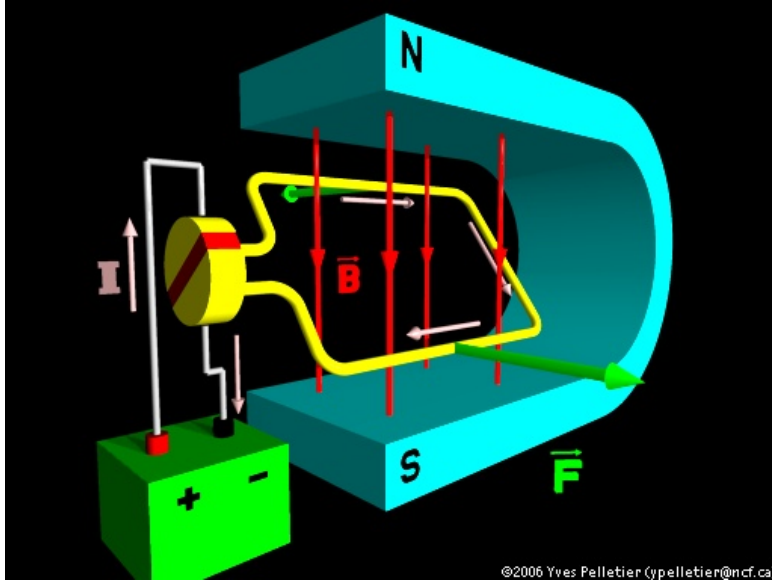


Figure 4: Electric motors is an important application of Faraday's Law.

The force \vec{F} on a wire of length $\vec{\ell}$ carrying a current I in a magnetic field \vec{B} is

$$\vec{F} = I\vec{\ell} \times \vec{B} = IlB\sin(\theta)$$

Where θ would be 90 if the field were uniformly vertical (which we assume for simplicity). The direction of \vec{F} comes from the right hand rule. The two forces shown here are equal and opposite, but they are displaced vertically, so they exert a torque, τ . (The forces on the other two sides of the coil act along the same line and so exert no torque.) A DC motor is also a DC generator. If you use mechanical energy to rotate the coil (N turns, area A) at uniform angular velocity ω in the magnetic field B , it will produce a sinusoidal electromotive force (emf) in the coil. Let θ be the angle between B and the normal to the coil, so the magnetic flux ϕ_B is $NAB \cdot \cos(\theta)$. Where θ is defined as $\theta = \omega t$. Faraday's law gives:

$$\begin{aligned} \mathcal{E} &= -\frac{d\phi_B}{dt} \\ &= -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \\ &= -\frac{d}{dt} [BA\cos(\theta)] \\ &= -BA\frac{d}{dt}\cos(\omega t) \\ &= BA\sin(\omega t)\frac{d(\omega t)}{dt} \\ \mathcal{E} &= NBA\omega\sin(\omega t) \end{aligned} \tag{11}$$

When we are dealing with a maximum (peak) voltage or emf then it simply becomes

$$\mathcal{E} = NBA\omega$$

3.2.1 Magnetic Dipole Moment

The moment arm for the torque is the height of the dipole, $L\sin(\theta)$. The angle θ is between the direction of the magnetic field and a vector normal to the plane of the dipole. Therefore the torque disappears when the dipoles orientation is along the axis of the magnetic field. We know that torque, τ , is due to the forces on the edge of the coil as pictured above in the electric motor. We see that the forces from both sides contribute some amount of torque. Assume the width of the coil is w and that the coil is square. The total area of coil would then be $A = wL$.

$$\begin{aligned}\tau &= \vec{r} \times \vec{F} + \vec{r} \times \vec{F} \\ \tau &= \frac{w}{2}ILB\sin(\theta) + \frac{w}{2}ILB\sin(\theta) \\ \tau &= wILB\sin(\theta) \\ \tau &= AIB\sin(\theta)\end{aligned}$$

If we define the magnetic dipole moment vector as

$$\vec{\mu} = NI\vec{A} \quad (12)$$

Where N is the number of coils, I is the current, and \vec{A} is the area vector, then we can define the torque on a current carrying loop as a vector product of the magnetic dipole moment and the magnetic field

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad (13)$$

The torque always tries to rotate the angle θ to zero. This is where the potential energy is a minimum. To find the potential energy of a magnetic dipole moment we use the work-energy principle for rotational motion

$$\begin{aligned}U &= \int \tau d\theta \\ &= \int NIAB\sin(\theta)d\theta \\ &= NIAB \int \sin(\theta)d\theta \\ &= -\mu B\cos(\theta) + C \longrightarrow C = 0 \\ U &= -\vec{\mu} \cdot \vec{B}\end{aligned} \quad (14)$$

We can arbitrarily choose the initial energy to be zero at $\theta = \frac{\pi}{2}$ so $C = 0$.

3.3 Transformers

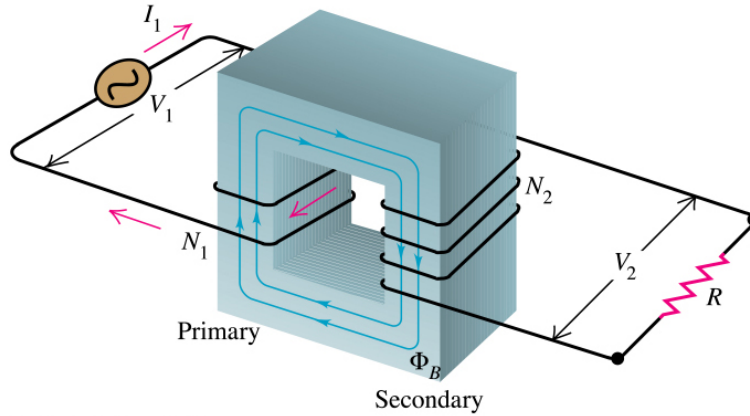


Figure 5: A standard transformer used in power line transmission and other applications for stepping down or up voltage.

The core has high magnetic permeability (ie a material that forms a magnetic field much more easily than free space does, due to the orientation of atomic dipoles). The result is that the field is concentrated inside the core, and almost no field lines leave the core. It follows that the magnetic flux through the primary and secondary are equal, as shown. From Faraday's law, the emf in each turn, whether in the primary or secondary coil, is $\frac{d\phi_B}{dt}$. If we neglect resistance and other losses in the transformer, the terminal voltage equals the emf. For the N_p turns of the primary, this gives

$$V_P = N_P \frac{d\phi_B}{dt}$$

And alternatively for N_s turns in the secondary coil we get

$$V_s = N_s \frac{d\phi_B}{dt}$$

Taking the ratios of the previous two equations we get the transformer voltage ratio

$$\frac{V_s}{V_P} = \frac{N_s}{N_P} \tag{15}$$

Since we cannot get extra power from nothing nor lose power (ideally) we can assume the powers are equal and find the different currents produced by $P = I_s V_s = I_P V_P$. Taking these ratios we get the transformer current ratio

$$\frac{I_s}{I_P} = \frac{N_P}{N_S} \tag{16}$$

3.4 Inductance and Inductor's

When dealing with current carrying coils there will always be some inductance. In electromagnetism and electronics, inductance is that property of a conductor by which a change in current in the conductor induces a voltage (electromotive force) in both the conductor itself

(self-inductance) and any nearby conductors (mutual inductance). Inductance, L , is defined as the magnetic flux divided by current, however, the inductance is purely dependent on the geometry of the coils. If we write Faradays law for some coils with $L = \frac{\phi_B}{I}$ we will obtain

$$\begin{aligned}\mathcal{E} &= -\frac{d\phi_B}{dt} \\ \mathcal{E} &= -\frac{d(LI)}{dt} \\ \mathcal{E} &= -L\frac{dI}{dt}\end{aligned}\tag{17}$$

This is the emf that will oppose the original emf, hence we call it the back emf. This back emf "impedes" the flow of charge.

3.5 Magnetic Field Energy Density

We know that power, P , is defines as voltage multiplied by current. When an inductor has current flowing through it it generates a emf given by $\mathcal{E} = -L\frac{dI}{dt}$. To find the energy stored in the inductors magnetic field we use the definition of power, $P = \frac{dU}{dt}$, hence the energy stored

$$\begin{aligned}dU &= P dt \\ \int dU &= \int P dt \\ U &= \int L\frac{dI}{dt} dt \\ U &= L \int IdI \\ U &= \frac{1}{2}LI^2\end{aligned}\tag{18}$$

I will omit derivation due to length of paper but don't hesitate to ask for derivation next time you see me. The energy density , u is just

$$u_B = \frac{B^2}{2\mu_0}\tag{19}$$

Compare this to the energy density stored in the electric field

$$u_E = \frac{1}{2}\epsilon_0 E^2\tag{20}$$

4 LRC and AC Circuits

There are many applications of AC Circuits. The way we analyze AC circuits is utilizing complex plane and complex numbers which we call phasors. I will omit the theory and derivation for this section and only state the results of the complicated mathematics.

4.1 LR Circuit

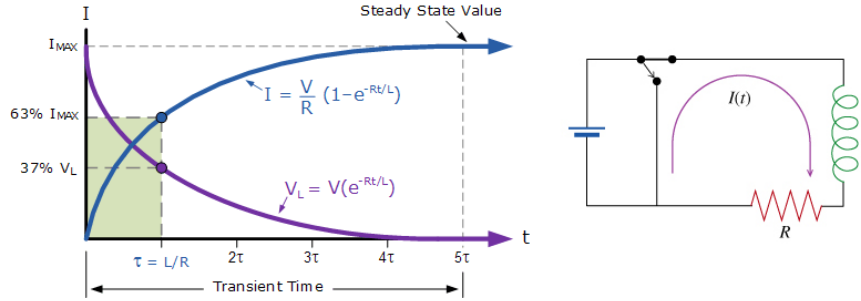


Figure 6: An LR circuit. Also, the graph of currents when emf and no emf is applied.

4.1.1 LR Circuit with EMF

If a circuit is switched on, the current cannot instantly rise to the value determined by its resistance because of the opposing emf produced by its inductance. If an emf \mathcal{E} , a resistance R , and an inductance L are connected together in series, the loop rules can be used to find the differential equation that determines the way the current rises from zero up to its final value

$$L \frac{dI}{dt} + IR = \mathcal{E}$$

The solution of this equation for a circuit with zero current at time $t = 0$ is

$$I(t) = I_0(1 - e^{-\frac{t}{L/R}}) \quad (21)$$

The quantity L/R is called the inductive time constant, or the L-over-R time, of the circuit; it plays the same role here that the time constant RC played in the resistor-capacitor circuit. Note that the equation for $I(t)$ is very similar to the equation for $Q(t)$ in a charging capacitor.

4.1.2 LR Circuit without EMF

It is not so simple to turn a circuit off. If we just open a switch and very quickly stop the current, Faraday's law says that an enormous emf should be produced because of the very short time involved. Very often this enormous emf is large enough to cause the air to break down and spark. This is just Lenz's law at work again, fighting the change in the current, and it is easy to produce this effect. Try turning the vacuum cleaner off by pulling the plug out of the wall, and you will see a flash at the outlet. This flash is caused by Lenz's law in the large inductance of the windings in the vacuum cleaner motor. To avoid a flash, it is necessary to design a special kind of switch that gives the current a conducting path through which to flow while it decreases. If this has been done, then the loop rules applied to a circuit with inductance L and resistance R give

$$L \frac{dI}{dt} + IR = 0$$

whose solution, for initial current I_0 , is given by

$$I(t) = I_0 e^{-\frac{t}{L/R}} \quad (22)$$

4.1.3 LC Circuit

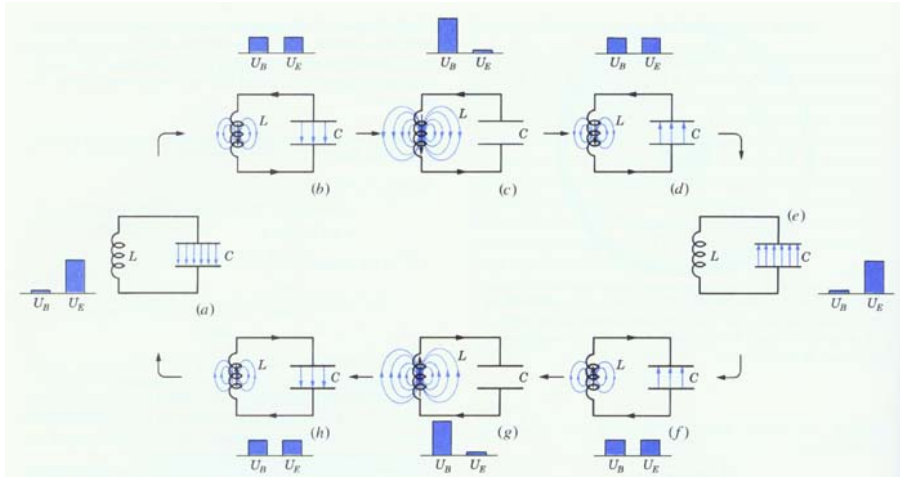


Figure 7: An LC circuit. The energy oscillates between being stored in the magnetic field of the inductor and the electric field of capacitor. That's why we call it an electromagnetic oscillator.

If a capacitor is allowed to discharge through a resistor, the charge just exponentially dies away. If a capacitor is allowed to discharge through an inductor, something completely different happens. Since an inductor stores energy, but does not dissipate it, the energy stored in the capacitor must simply be transferred to the inductor during the discharge. The result is an oscillation with the energy repeatedly passed back and forth between the capacitor and the inductor.

The equation that describes the way the charge on the capacitor varies with time is the following second order differential equation, which may be obtained from the Kirchoff's loop theorem in the usual way

$$L \frac{d^2 Q}{dt^2} + \frac{Q}{C} = 0$$

The solution of this equation corresponding to having charge Q_0 on the capacitor at the beginning of the oscillation is

$$Q(t) = Q_0 \cos(\omega t) \quad (23)$$

Where the angular frequency of the oscillation is given by

$$\omega = \frac{1}{\sqrt{LC}}$$

The current in the circuit can either be obtained from $I = -\frac{dQ}{dt}$, or from conservation of energy

$$I(t) = Q_0 \omega \sin(\omega t) \quad (24)$$

4.2 LRC Circuit

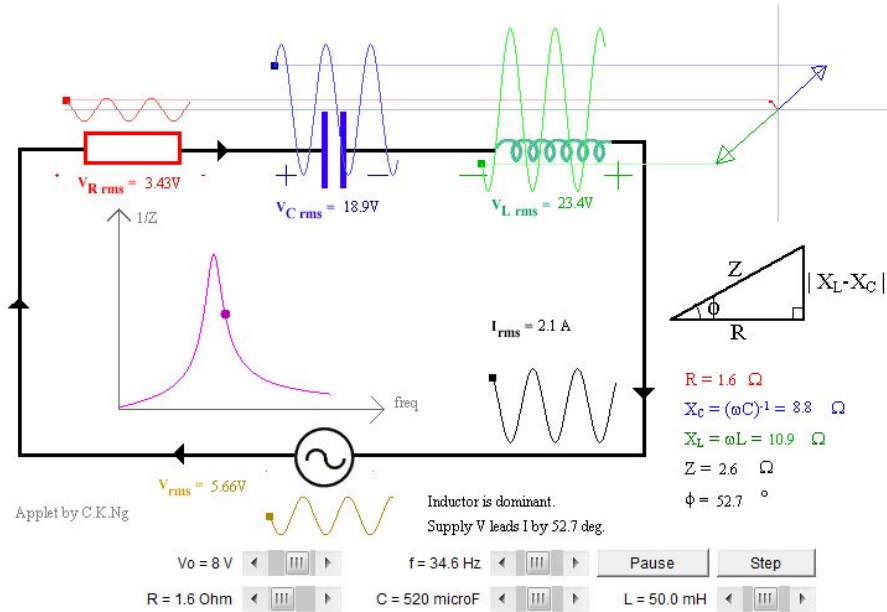


Figure 8: An LRC circuit. This picture was a graph made in an MathCAD program and shows the phasors corresponding the sinusoidal voltages of the circuit elements which are out of phase.

The formulas for the LC oscillator assume that there is no resistance. But, of course, there is almost always resistance, so we need to know how the oscillations change when resistance is included. The simplest way to see what happens is to use an analogy between the electric circuit and an oscillating mass-spring system. The mass is like the inductance, the spring is like the capacitance, and friction is like resistance.

If the friction is very large (imagine putting the oscillating mass in a bucket of honey) and if the spring is not very strong, then the velocity of the mass will be so small that its inertia never becomes important, and the system won't oscillate. If the mass is pulled down, it just slowly returns to its equilibrium position. This is exactly what happens in an RLC circuit if the resistance is very large; the capacitor just slowly discharges through the resistor without any oscillation.

If the friction is very small (imagine letting the mass move in the air) then the mass will oscillate freely, but with a slowly decreasing amplitude. This is also the behavior of the RLC circuit with small resistance; the charge on the capacitor oscillates many times, but the maximum charge on the capacitor slowly decreases with time. We call such an oscillation a damped oscillation.

The differential equation that gives $Q(t)$ for a circuit containing a capacitor C , an inductor L , and a resistor R is

$$L \frac{d^2 Q}{dt^2} + \frac{dQ}{dt} R + \frac{Q}{C} = 0$$

In the case of small enough resistance that oscillations occur, the solution of this equation is

$$Q(t) = Q_0 e^{-\frac{t}{2L/R}} \cos(\omega_d t) \quad (25)$$

where the oscillation frequency, ω_d , is modified from $\omega = \frac{1}{\sqrt{LC}}$ by the resistance

$$\omega_d = \sqrt{\frac{1}{\sqrt{LC}} - \frac{R}{2L^2}} \quad (26)$$

Note that if the resistance becomes too large, the argument of the square root in this formula becomes negative. This indicates that the circuit has changed its behavior from damped oscillations to damping without oscillations, but the new formulas will not be given here. You will encounter them if you study differential equations.

4.3 AC Circuits

The oscillations we have discussed up to now are free oscillations in which the system is given some energy, then left alone. For instance, you could pull a child on a swing up to a certain height, then let go and wait for the motion to die away. But this is not the only possibility; we could also repeatedly push the swing at any frequency we like and watch what happens. In this case we say that we have forced oscillations. There are now two frequencies in the problem: the natural frequency of the free oscillations, and the driving frequency of the forced oscillations. This means that you will have to resist the urge to use the formula $\omega = \frac{1}{\sqrt{LC}}$ whenever you encounter a frequency. If the frequency in question is the driving frequency, there is no formula for it; it is simply set by the design of the driving circuit.

4.3.1 Capacitor Circuit

If the voltage applied is given by

$$V_C t = V_0 \sin(\omega t)$$

The resulting current through the capacitor is given by

$$I_C(t) = \omega C V_0 \cos(\omega t) \quad (27)$$

We now try to make this situation similar to the resistor circuit by insisting that the magnitude of the current be given by a formula that looks like Ohm's law, $I_C = \frac{V_L}{X_C}$ with

$$X_C = \frac{1}{\omega C} \quad (28)$$

The quantity is called the X_C , capacitive reactance, has units of ohms, and is used just like a resistance to find the magnitude of the current that flows due to an applied voltage.

Note that the applied voltage and the resulting current are not given by the same function of time, meaning that they are out of phase with each other. If we start watching this circuit at time $t = 0$, then we see the current at its maximum value and the applied voltage at zero. A quarter of a cycle later the current has dropped to zero and the voltage has risen to its maximum value. We describe this situation by saying that the current is ahead of the voltage, or that it leads the voltage. Sometimes we also say that the voltage is behind, or lags, the current. It may be helpful to remember the word ICE for this circuit; the C stands for capacitor, and in "ICE" the current, I, is ahead of the voltage, or applied emf, \mathcal{E} .

4.3.2 Inductive Circuits

If the voltage applied is given by

$$V_L t = V_0 \sin(\omega t)$$

The resulting current through the inductor is given by

$$I_L(t) = \frac{V_0}{\omega L} \cos(\omega t) \quad (29)$$

We now try to make this situation similar to the resistor circuit by insisting that the magnitude of the current be given by a formula that looks like Ohm's law, $I_L = \frac{V_0}{X_L}$ with

$$X_L = \omega L \quad (30)$$

The quantity is called the inductive reactance, has units of ohms, and is used to find the magnitude of the current that flows due to an applied voltage.

Again, the applied voltage and the resulting current are not given by the same function of time, so they are out of phase. If we start watching this circuit at time $t=0$, then we see the current at its minimum (negative) value and the applied voltage at zero. A quarter of a cycle later the current has made it up to zero on its way up to its maximum, but the voltage has already risen to its maximum value. We describe this situation by saying that the voltage is ahead of, or leads, the current by 90° . It may be helpful to remember the word ELI for this circuit; the L stands for inductor, and in ELI the voltage, or emf E, comes before the current, I. To remember both the capacitor and the inductor words, remember "ELI the ICE man" who used to help keep things cold until the engineers who developed these circuits put him out of business.

5 References

I used many sources such as Wikipedia, MIT's OCW, SJSU Lecture slides, Giancoli Textbook, etc to make this document as thorough as possible.