Physics 116A Practice Midterm

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1. Fnd the limit of the given sequence as $n \to \infty$

$$\frac{(n+1)^2}{\sqrt{3+5n^2+4n^4}} \tag{1}$$

Solution: The first thing to do in these problems is factor out the highest order of n on the denominator. Then see where they cancel. Here we see that the numerator has highest order of n^2 and the denominator is $\sqrt{n^4} = n^2$ so the n dependence should cancel after factoring (at least in numerators).

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)^2}{\sqrt{3+5n^2+4n^2}} &= \lim_{n\to\infty} \frac{n^2(1+1/n)^2}{n^2\sqrt{3/n^4+5/n^2+4}} \\
&= \lim_{n\to\infty} \frac{(1+1/n)^2}{\sqrt{3/n^4+5/n^2+4}} \\
&= \frac{(1+1/\infty)^2}{\sqrt{3/\infty^4+5/\infty^2+4}} \\
&= \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}
\end{aligned}$$

Notice that it turned out that the result was just what the coefficients of the highest order terms were. Be sure to be on the look out for simple limits like this on the GRE. It can save you tons of time and improve your score tremendously.

2. Find the interval of convergence of:

$$\sum_{n=1}^{\infty} \frac{(-2)^n(2x+1)^n}{n^2} \tag{2}$$

be sure to investigate the endpoints of the interval.

Solution: When finding the interval of convergence we just do the ratio test and see for what values of x does the series converge. Or any other words $\rho < 1$

$$\begin{aligned}
\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(2x+1)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-2)^n(2x+1)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-2)(2x+1)}{(n+1)^2}\cdot\frac{n^2}{1}\right| \\
&= \lim_{n\to\infty} 2\,|2x+1|\left|\frac{n}{n+1}\right|^2 \\
&= \lim_{n\to\infty} 2\,|2x+1)|\left|\frac{n}{n(1+1/n)}\right|^2 \\
&= \lim_{n\to\infty} 2\,|2x+1|\left|\frac{1}{(1+1/n)}\right|^2 \\
&=2\,|(2x+1)|
\end{aligned}$$

Now this has to be less than one (I divided the two on both sides)to converge hence,

$$|2x+1| < 1/2 \longrightarrow -3/4 < x < -1/4$$

Now we must investigate the endpoints. If we plug in $x = -3/4$ we get

$$\sum_{n=1}^{\infty}\frac{(-2)^n(2(-3/4)+1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-2)^n}{(-2)^n n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This does converge by p-test.
Now for $x = -1/4$,

$$\sum_{n=1}^{\infty}\frac{(-2)^n(2(-1/4)+1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-2)^n(-1/2)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

So this converges as well by the p-test. The answer is then

$$\boxed{-3/4 \le x \le -1/4}$$

3. Find the following limits using Maclaurin series
   - A. $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$
   - B. $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos x}{\sin^2 x}\right)$
   - C. $\lim_{x\to 0}\left(\csc^2 x - \frac{1}{x^2}\right)$
   - D. $\lim_{x\to 0}\left(\frac{ln(1+x)}{x^2} - \frac{1}{x}\right)$

Solution: Since we are dealing with the Maclaurin series we only need to Taylor expand to the first order.
Solution to Part A)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{(1+x+x^2/2+O(x^3))-1}\right)$$

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)&=\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{(x+x^2/2+O(x^3))}\right)\\
&\approx\lim_{x\to 0}\left[\frac{1}{x}\left(1-\frac{1}{1+x/2+O(x^2)}\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{1}{x}\left(1-(1-x/2+O(x^2))\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{1}{x}\left(x/2-O(x^2)\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{x}{x}\left(1/2-O(x^1)\right)\right]\\
&=\boxed{1/2}
\end{aligned}$$

Solution to Part B)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos x}{\sin^2 x}\right)&=\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1-O(x^2)}{(x-x^3/6+O(x^5))^2}\right)\\
&\approx\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1-x^2/2+O(x^4)}{(x^2-2x^4/6+O(x^6))}\right)\\
&\approx\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1-x^2/2+O(x^2)}{x^2(1-x^2/3+O(x^4))}\right)\\
&\approx\lim_{x\to 0}\left[\frac{1}{x^2}\left(1-\frac{1-x^2/2+O(x^2)}{(1-x^2/3+O(x^4))}\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{1}{x^2}\left(\frac{1-x^2/3+O(x^4)}{1-x^2/3+O(x^4)}-\frac{1-x^2/2+O(x^2)}{(1-x^2/3+O(x^4))}\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{1}{x^2}\left(\frac{1-x^2/3+O(x^4)-(1-x^2/2+O(x^4))}{1-x^2/3+O(x^4)}\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{1}{x^2}\left(\frac{-x^2/3+x^2/2+O(x^4))}{1-x^2/3+O(x^4)}\right)\right]\\
&\approx\lim_{x\to 0}\left[\frac{1}{x^2}\left(\frac{x^2/6+O(x^4)}{(1-x^2/3+O(x^4)}\right)\right]\\
&\approx\lim_{x\to 0}\left(\frac{1/6+O(x^2)}{(1-x^2/3+O(x^4)}\right)\\
&=\boxed{1/6}
\end{aligned}$$

Solution to Part C)

$$
\begin{aligned}
\lim_{x\to 0}\left(\csc^2 x - \frac{1}{x^2}\right) &= \lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) \\
&= \lim_{x\to 0}\left(\frac{1}{(x - x^3/6 + O(x^5))^2} - \frac{1}{x^2}\right) \\
&= \lim_{x\to 0}\left(\frac{1}{(x^2 - 2x^4/6 + O(x^6))} - \frac{1}{x^2}\right) \\
&= \lim_{x\to 0}\left(\frac{1}{(1 - x^2/3 + O(x^4))} - 1\right)\frac{1}{x^2} \\
&\approx \lim_{x\to 0}\left((1 + x^2/3 - O(x^4)) - 1\right)\frac{1}{x^2} \\
&\approx \lim_{x\to 0}\left(x^2/3 + O(x^4))\right)\frac{1}{x^2} \\
&\approx \lim_{x\to 0}\left(1/3 + O(x^2))\right)\frac{x^2}{x^2} \\
&\approx \lim_{x\to 0}\left(1/3 + O(x^2))\right) \\
&\approx \boxed{1/3}
\end{aligned}
$$

Solution to Part D)

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{ln(1+x)}{x^2} - \frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{x - x^2/2 + O(x^3)}{x^2} - \frac{1}{x}\right) \\
&\approx \lim_{x\to 0}\left(\frac{1 - x/2 + O(x^2)}{x} - \frac{1}{x}\right) \\
&\approx \lim_{x\to 0}\left(1 - x/2 + O(x^2) - 1\right)\frac{1}{x} \\
&\approx \lim_{x\to 0}\left(-x/2 + O(x^2)\right)\frac{1}{x} \\
&\approx \lim_{x\to 0}\left(-1/2 + O(x)\right)\frac{x}{x} \\
&\approx \lim_{x\to 0}\left(-1/2 + O(x)\right) \\
&= \boxed{-1/2}
\end{aligned}
$$

Make sure you expand to correct to order. If you don't all these results will get you something of the nature $1/x - 1/x \neq 0$. I made this mistake. Don't be like me. Also, we made use of binomial expansion $(1+x)^n \approx 1 + nx + n(n-1)x^2/2 + \cdots$. This is used a lot so remember it. I believe Newton said the general binomial expansion is the greatest contribution he made. So it must be good!

4. Find the disk of convergence for the following complex power series.

$$
\sum_{n=0}^{\infty} \frac{(n!)^3 z^n}{(3n)!} \tag{3}
$$

Solution: We use the ratio test and find values of z which make $\rho < 1$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)!)^3(z)^{n+1}}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^3(z)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{((n+1)!)^3(z)}{(3n+3)!}\cdot\frac{(3n)!}{(n!)^3}\right|$$
$$= \lim_{n\to\infty}\left|\frac{((n+1)!)^3}{(n!)^3}\cdot\frac{(3n)!}{(3n+3)!}\right||z|$$

So now we have two tricky parts, the two factorial pieces. Let us handle them separately.

$$\left|\frac{((n+1)!)^3}{(n!)^3}\right| = \left|\frac{(n+1)!}{(n!)}\right|^3$$
$$= \left|\frac{(n+1)(n)(n-1)\cdots}{n(n-1)\cdots}\right|^3$$
$$= |n+1|^3$$
$$= \left|n^3(1+1/n)^3\right|$$

Now let us take care of the second one.

$$\left|\frac{(3n)!}{(3n+3)!}\right| = \left|\frac{(3n)(3n-1)(3n-2)(3n-3)\cdots}{(3n+3)(3n+2)(3n+1)(3n)(3n-1)\cdots}\right|$$
$$= \left|\frac{1}{(3n+3)(3n+2)(3n+1)}\right|$$
$$= \left|\frac{1}{n(3+3/n)n(3+2n)n(3+1/n)}\right|$$
$$= \left|\frac{1}{n^3(3+3/n)(3+2/n)(3+1/n)}\right|$$

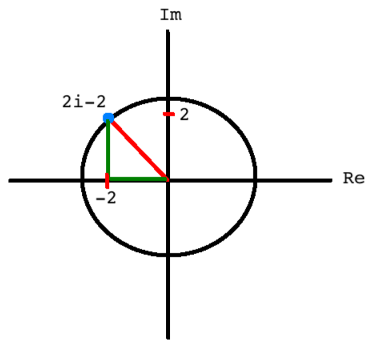
Putting all of this stuff together we get

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n^3(1+1/n)^3}{n^3(3+3/n)(3+2/n)(3+1/n)}\right||z|$$
$$= \lim_{n\to\infty}\left|\frac{(1+1/n)^3}{(3+3/n)(3+2/n)(3+1/n)}\right||z|$$
$$= \left|\frac{1}{27}\right||z| \longrightarrow \boxed{|z| < 27}$$

5. Find all values of the roots of

$$\sqrt[3]{2i-2} \tag{4}$$

Solution: Remember that any complex number can be represented in polar form, $re^{i\theta}$, (often the best way to represent complex numbers) as can be seen in the figure below.



We need to know what the radius r is, and what the and what angle θ is.

$$r = |z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

and what the and what angle θ is,

$$\theta = \arctan\left(\frac{2}{-2}\right) = \pi/4 + \pi/2 = 3\pi/4$$

Now here is the important part. The angle $\pi/4 = (\pi/4 + 2\pi n)\ \forall\ n \ \exists\ \mathbb{Z}$. This is due to the nature of complex numbers as having a length AND a sinusoidal phase. Putting this all together we get

$$2i - 2 = \sqrt{8}e^{i(\pi/4+2\pi n)}$$

Taking the cube root of that we get

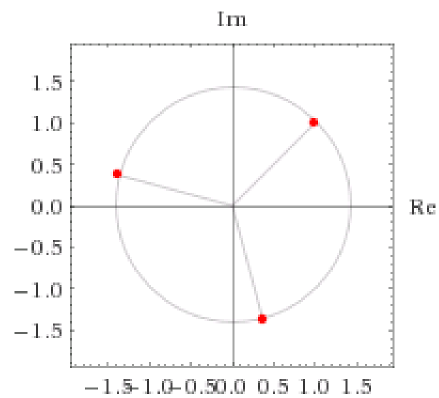
$$\begin{aligned}\sqrt[3]{2i-2} &= \left(\sqrt{8}e^{i\pi(3/4+2n)}\right)^{1/3} \\ &= \sqrt[6]{8}e^{i\pi(3/12+8n/12)} = \sqrt{2}e^{i\pi(3/12+8n/12)}\end{aligned}$$

Ok so now we must check what values of n give us argument values $-\pi < argz < \pi$, because values outside that range will give redundant information.

n	$\pi(3/12 + 8n/12)$	$< \lvert\pi\rvert$
0	$3\pi/12 = \pi/4$	✓
1	$11\pi/12 = 3\pi/4$	✓
2	$19\pi/12$	✗
-1	$-5\pi/12$	✓
-2	$-15\pi/12$	✗

So the principal values of the roots are for $n = 0, \pm 1$ which gives $x + iy$ forms

$$\boxed{x + iy = \sqrt{2}(1+i), -1.3660 + 0.3660i, 0.3660 - 1.3660i}$$



There are three roots just as we would expect for a cube root.

6. Evaluate each of the following in $x + iy$ form
   A. $(-i)^{\sin i}$
   B. $\cos(2i \ln i)$

Solution: Solution to Part A
Rewrite $-i$ in polar form: $-i = e^{-i\pi/2}$. Then we use the relation between sine and hyperbolic sine

$$\sin(ix) = i \sinh(x)$$

$$(-i)^{\sin(i)} = (e^{-i\pi/2})^{i \sinh(1)}$$
$$= \boxed{e^{\pi \sinh(1)/2}}$$

Solution to Part B
First thing we need to do is handle the natural log, $\ln i = \ln e^{i()\pi/2+2\pi n)} = i\pi(1/2 + 2n)$. Therefore,

$$\boxed{\cos(2i \ln i) = \cos((2i)i\pi(1/2 + 2n)) = \cos(-\pi + 2\pi n) = 1}$$

It equals 1 for all n values.

7. Express the following integrals as a Γ function.

$$\int_0^1 \sqrt[3]{\ln x} dx \tag{5}$$

Solution: First thing is to use u-sub[1]. Let us try $u = -\ln x \longrightarrow x = e^{-u} \longrightarrow dx = -e^{-u}du$. The limits are now $u(0) = -\ln(0) = \infty$ and $u(1) = -\ln(1) = 0$. Now are integral is

$$
\begin{aligned}
\int_0^1 \sqrt[3]{\ln x}dx &= \int_\infty^0 \sqrt[3]{-u}(-e^{-u})du \\
&= -\int_0^\infty (-1)^{4/3}u^{1/3}e^{-u}du \\
&= -\int_0^\infty u^{1/3}e^{-u}du \\
&= \boxed{-\Gamma(4/3)}
\end{aligned}
$$

Where I have (questionably) done away with the negative 3rd root by

$$-(-u)^{1/3} = (-1)^{3/3}(-1)^{1/3}u^{1/3} = (-1)^{4/3}u^{1/3} = [(-1)^4]^{1/3}u^{1/3} = u^{1/3}$$

8. Evaluate the integral as a B, and then express in terms of Γ functions.

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} \tag{6}$$

Solution: Let us use y-substitution with $y = x^3$ and then $dx = (1/3)x^{-2/3}dx$. The limits stay the same. Replacing this with our given integral we get

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} =(1/3)\int_0^1 \frac{y^{-2/3}dy}{(1-y)^{1/2}}$$

The form of B functions are given one way by Boas Eq. 14.6.1:

$$B(p,q) = \int_0^1 \frac{y^{p-1}dy}{(1-y)^{p-1}}$$

Which is the form we have so all we have to do is find the p and q values that give us our integral. For p we have

$$p - 1 = -2/3 \longrightarrow p = 1/3$$

and for q we have

$$q - 1 = 1/2 \longrightarrow q = 1/2$$

Therefore we have the

$$\boxed{\int_0^1 \frac{dx}{\sqrt{1-x^3}} = (1/3)B(1/3, 3/2)}$$

Which by Boas Eq. 11.7.1 we get

$$\boxed{(1/3)B(1/3, 1/2) = (1/3)\frac{\Gamma(1/3)\Gamma(1/2)}{\Gamma(5/6)} = \frac{\Gamma(1/3)\sqrt{\pi}}{3\Gamma(5/6)}}$$