Proof of Katznelson's Theorem

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Abstract

This paper sets out to prove a theorem proposed by Dr. Yonatan Katznelson. I will prove Katznelson's theorem using only basic integer, $\mathbb{Z}$, properties.

1 Katznelson's Theorem

Theorem 1.1. *For every integer n, there is a unique pair of integers k and l satisfying*

- $0 \leq l \leq 2$ *and*
- $n = 3k + l$.

2 Properties of Integers

Arithmetic properties

1. $\mathbb{Z}$ is closed under addition and multiplication.
2. Addition and multiplication are both commutative and associative, and multiplication distributes over addition.
3. Every integer a has an additive inverse $-a$ satisfying $a + (-a) = 0$.

Properties related to the natural order ($>$) on $\mathbb{Z}$

4. If $n > 0$ and $a \in \mathbb{Z}$, then $a + n > a$.
5. If $n > 0$, then $-n < 0$.
6. If $n > 0$ and $m > 0$, then $n \cdot m > 0$. If $n < 0$ and $m < 0$, then $n \cdot m > 0$. If $n > 0$ and $m < 0$, then $n \cdot m < 0$.

3 Proof

Proof. Let there exist $a, b, c, d \in \mathbb{Z}$ such that for all values of n there is a **unique** pair of integers $\{a, b\}$ that satisfy Katznelson's Theorem and a **theoretical** pair of integers $\{c, d\}$ that also satisfy Katznelson's Theorem. *Assume:*

1. $n = 3a + b$

2. $n = 3c + d$

If Katznelson's Theorem is false then both equations 1 and 2 can be true for the same value of n.

Let:

$$3a + b = 3c + d$$

Then:

$$3a - 3c + b - d = 0 \quad (1)$$
$$3(a - c) + (b - d) = 0 \quad (2)$$
$$a = c \quad (3)$$
$$b = d \quad (4)$$

It is obvious to see that (3) and (4) must be true for two separate pairs of integers to satisfy Katznelson's Theorem, consequently (3) and (4) show that both pairs are identical. However, we can also produce two separate unique pairs of integers if we change only one value of the pair by allowing $a = c$ or $b = d$. We have the separate integer pairs in two different situations $\{a, b\}$,$\{c, b\}$ and $\{a, b\}$,$\{a, d\}$. *Assume:*

1. $n = 3a + b$

2. $n = 3c + b$

3. $n = 3a + b$

4. $n = 3a + d$

Relationships 1 through 4 all have unique pairs of integers satisfying Katznelson's Theorem, yet we see that all pairs must be exactly equivalent by setting 1 equal to 2 and 3 equal to 4.

$$3a + b = 3c + b \quad (5)$$
$$3a = 3c \quad (6)$$
$$a = c \quad (7)$$

Similarly,

$$3a + b = 3a + d \quad (8)$$
$$b = d \quad (9)$$

From both situations presented we can conclude that there is only one unique pair of integers $k, l \in \mathbb{Z}$ for each value of n. □